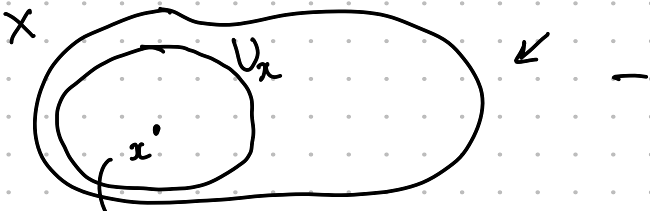
Regular functions

- Local definition.

Prop: For $X \subset \mathbb{A}^n$ closed

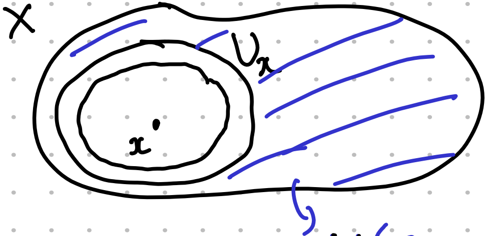
regular $\iff$ polynomial.

Pf: $f : X \to k$ regular $\qquad \mathbb{A}^n$



$f = \dfrac{p_x}{q_x}$ $\quad q_x(y) \neq 0$
$\forall y \in U$

① control p_x & q_x outside U_x. → make zero.



$V(g_1, \dots, \underbrace{\dots}_{g's})$

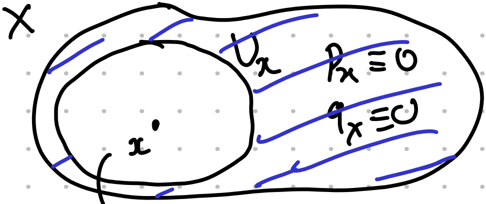
$\cap$

$V(g)$

$\exists$ g such that $g(x) \neq 0$

→ $x \in \underline{U'_x} = V(g)^c \subset U_x$

$\underline{p'} = p \cdot g$ & $\underline{q'} = q \cdot g$ ← zero outside U'_x.



$$f = \frac{p_x}{q_x}$$

$$X = V(I)$$

$$\boxed{\{\, g = 0 \mid g \in I,\ q_x = 0 \mid x \in X \,\}}$$

X $\qquad\qquad q_x(x) \neq 0.$

Nullstellensatz $\Rightarrow$

$$\underline{1 = \underbrace{g}_{I} + r_1 q_{x_1} + \cdots + r_m q_{x_m}} \quad (*)$$

$\circledast \Rightarrow \forall\ x \in X\ \ \exists\ i$:

$q_{x_i}(x) \neq 0$.

$\Rightarrow X = U_{x_1} \cup \ldots \cup U_{x_m}$

$P = r_1 P_{x_1} + \cdots + r_m P_{x_m}$

then $P = f$ on X.

<u>Why?</u> $P(x) = f(x) \quad \forall x \in X$

Suppose $x \in U_{x_1}$ & <u>no</u> <u>others</u>.

$P(x) = r_1 P_{x_1}(x)$

& $r_1 q_{x_1}(x) = 1$

$f = \dfrac{P_{x_1}}{q_{x_1}}$ on U_{x_1}

so $f(x) = P(x)$.

Suppose

$x \in U_{x_1}, \ldots, U_{x_i}$ & not in rest.

$$\underline{\underline{f(x)}} = \boxed{\frac{p_{x_1}(x)}{q_{x_1}(x)} = \cdots = \frac{p_{x_i}(x)}{q_{x_i}(x)}}$$

$$1 = \boxed{r_1 q_{x_1}(x) + \cdots + r_i q_{x_i}(x)}$$

$$P(x) = \boxed{r_1 P_{x_1}(x) + \cdots + r_i P_{x_i}(x)}$$

Property of fractions :—

If $\frac{a}{b} = \frac{c}{d}$ then $\frac{\lambda a + \mu c}{\lambda b + \mu d}$

also equal to this value. $\square$

Regular maps & affines

$\varphi: X \to Y$ induces

$\varphi^*: k[Y] \to k[X]$

want $f = \varphi^*$

Conversely for affine. Given

$f: k[Y] \to k[X]$ $\exists!\ \varphi: X \to Y$.

$\underline{X} \subset \mathbb{A}^n$, $\underline{Y} \subset \mathbb{A}^m = \{(y_1, \dots, y_m)\}$

Try to define $\underline{\underline{\varphi}}$ that induces f.

Suppose $\underline{\varphi} = (\underline{\varphi_1, \dots, \varphi_m})$

$$\boxed{\varphi^*(\underline{\underline{y_i}}) = \varphi_i}$$

Pull back:

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[dr, "\varphi^*(\lambda)"'] & Y \arrow[d, "\lambda"] \\ & k \end{tikzcd}$$

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[dr, "\varphi^*(y_i)"'] & (\varphi_1(x), \dots, \varphi_m(x)) = (y_1, \dots, y_m) \arrow[d, "\lambda"] \\ & \underline{\varphi_i(x)} \quad y_i \end{tikzcd}$$

Must take $\varphi_i = f(y_i)$

$\varphi : X \to \mathbb{A}^m$

why $\varphi : X \to Y$?

$I = I(Y)$

$$k[Y] = \frac{k[y_1, \dots, y_m]}{I(Y)}$$

To check $\varphi(X) \in Y$, suffices that $\varphi(x)$ satisfies all eq's

$$g(\varphi(x)) = 0 \quad \forall g \in I.$$

$$g(\varphi_1(x), \dots, \varphi_m(x)) =$$

$$g(\varphi_1, \dots, \varphi_m)(x) =$$

$$g(f(y_1), \dots, f(y_m))(x)$$

f is a $\underline{\underline{k}}$-alg. hom.

so
$$g(f(y_1), \dots, f(y_m))(x)$$
$$= \underbrace{f(g(y_1, \dots, y_m))}(x)$$

$\boxed{X}$

$g(y_1, \dots, y_m) \in I(Y)$

$= 0$ in $k[Y]$

$f(g) = 0$ in $\underline{\underline{k[X]}}$

$= 0.$

Why is $\varphi^* = f$!!

generate $k[Y]$

True by const on $y_1, \dots, y_m$

$\mathbb{A}^1 \to \mathbb{A}^2 = \quad t \mapsto (\underline{t^2}, \underline{t^3})$

$\varphi : t \mapsto (x,y) \qquad x = \underline{\underline{t^2}},\ y = \underline{\underline{t^3}}$

$\mathbb{A}^1 \to V(y^2 - x^3) \subset \mathbb{A}^2$

$X \xrightarrow{\varphi} Y$ bijection.

<u>NOT</u> <u>AN</u> <u>ISO</u> !

$\| \quad k[Y] \longrightarrow k[X]$ <u>not iso</u>

$\dfrac{k[x,y]}{(y^2 - x^3)} \xrightarrow{\varphi^*} k[\underline{t}]$ $\Bigg\}$ not <u>iso</u>

$\left| \begin{array}{l} x \longmapsto t^2 \\ y \longmapsto t^3 \end{array} \right\|$

<u>Last</u>: $\mathbb{A}^n \supset \{f \neq 0\} = U$

- Then U is isomorphic to a closed subset of $\mathbb{A}^{n+1}$

$$V \subset \mathbb{A}^{n+1} = \{(x_1, \dots, x_n, y)\}$$

$$\begin{tikzcd}
V(y \cdot f(x_1, \dots, x_n) - 1) \arrow[d, shift right=4, "\wr\wr"'] \arrow[d, shift left, "\text{forget } y"] \\
U \arrow[u, shift left=6, "y = \frac{1}{f(x_1, \dots, x_n)}"']
\end{tikzcd}$$

Ex:

$$\boxed{\mathbb{A}^1 \setminus \{0\}} = \{t \neq 0\}$$

$$\Big\| 2$$

$$\boxed{V(ty-1)} \subset \mathbb{A}^2$$

$$k[\mathbb{A}^1 \setminus \{0\}] \cong \frac{k[t,y]}{(ty-1)}$$

$$I(V(ty-1)) = (ty-1).$$ ← prime ideal & hence radical.

$$I(V(I)) = \sqrt{I}$$