Algebraic Geometry — Aug 6

(i) Zariski Topology.

Finite unions — do union of two

$X = V(A) \qquad Y = V(B)$

$X \cup Y = V(\ldots\ldots)$

$= V(\{ab \mid a \in A,\ b \in B\})$

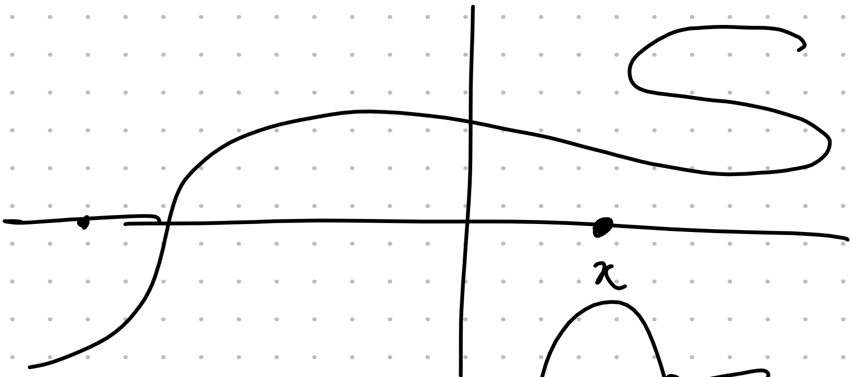
Caution — Infinite unions are not closed

eg. $\mathbb{A}^1$. Closed subsets are finite

$\bigcup_{\text{int}}$ (finite) could be infinite

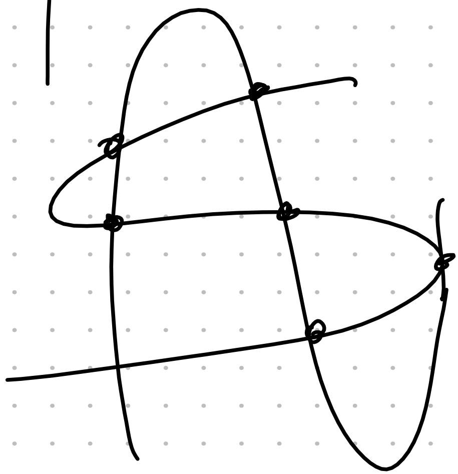
Pictures - What do Zariski closed subsets "look like"?

$V(f) \subset \mathbb{A}^2$ $f(x,y) = 0$ "curve"



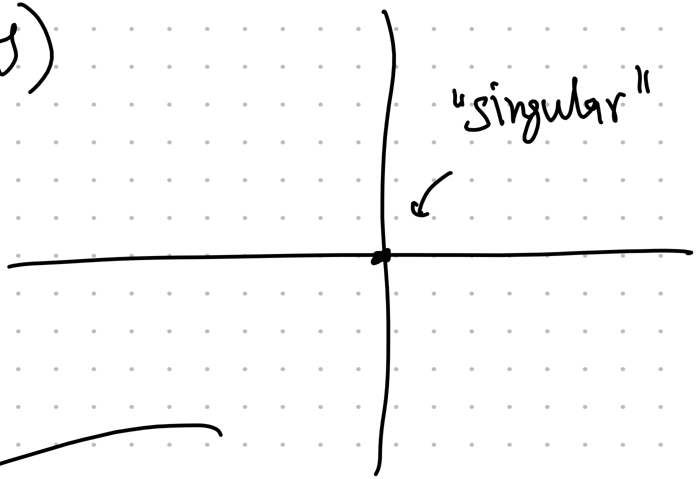
$\mathbb{A}^2_{\mathbb{C}}$

$V(f,g) =$

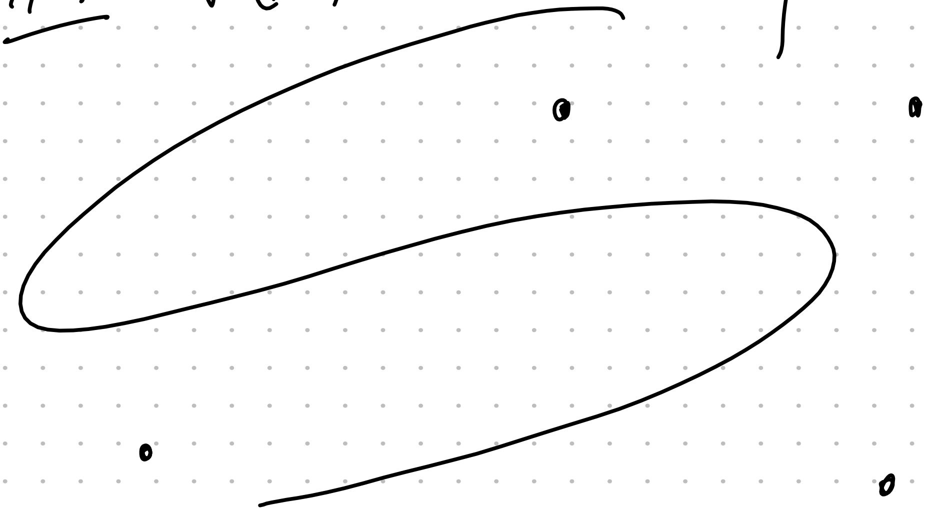


Is the set $V(f)=0$ a manifold?

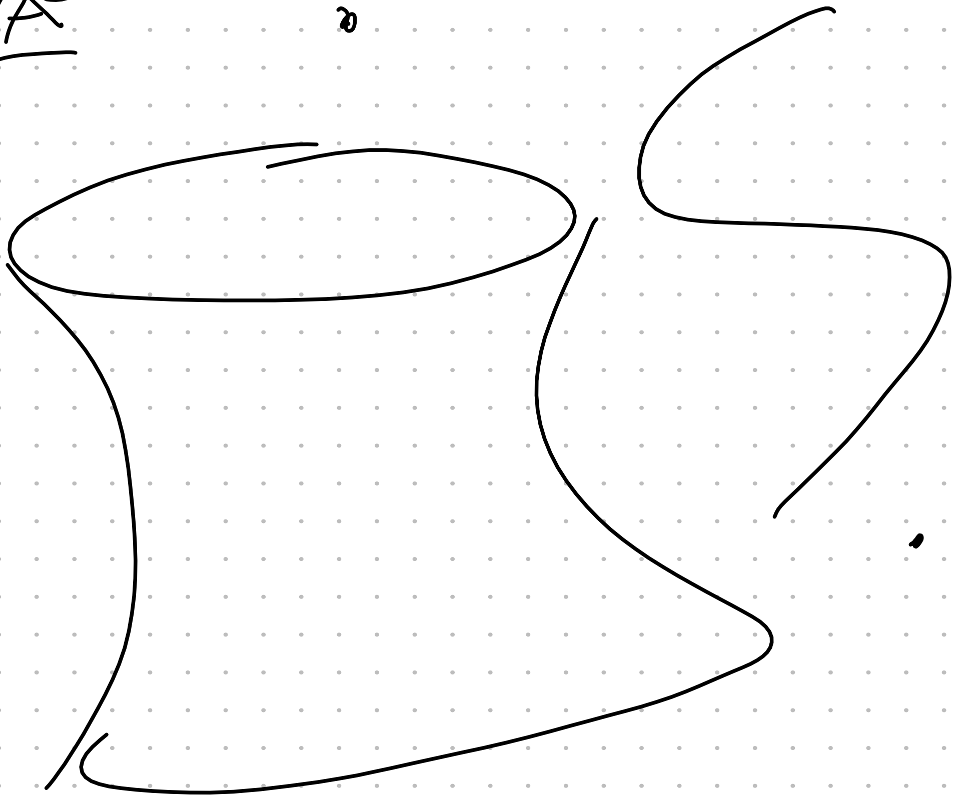
No. $V(xy)$



$\mathbb{A}^2$ $V(\mathbb{A})$



$\mathbb{A}^3$



(2) (3) Radical ideals & V's.

$\sqrt{I}$ an ideal. so closed under addition

$f \in \sqrt{I}$ $\Rightarrow$ $f^{n} \in I$

$g \in \sqrt{I}$ $g^{m} \in I$

$(f+g)^{n+m} \in I.$ $\sqrt[2]{I}$

$\sqrt[?]{I}$ $\sqrt[3]{I}$

(4) (5) (6) → Proof of Nullstellensatz

$$\{\text{Radical ideals}\} \underset{I}{\overset{V}{\rightleftarrows}} \{\text{Zariski closed}\}$$

↳ "Algebra" ↳ "Geometry"

Step 1. Maximal ideals $\mathfrak{m}$

$$k \longrightarrow k[x_1,\dots,x_n] \longrightarrow \underbrace{k[x_1,\dots,x_m]/\mathfrak{m}}_{L}$$

// $\boxed{k \longrightarrow L}$ "field ext^n"

Workhorse thm (Black box)

$\underline{k \to L}$ field ext., if L is fin. gen as a k-algebra ①, then L is fin dim as k-vspace ②

① $\exists$ finitely many $\underline{l_1, \dots, l_m} \in L$ s.t. every $l \in L$ can be written as $l =$ Poly. in $l_1, \dots, l_m$ coeff in $\underline{\underline{k}}$.

② $l =$ Linear exp in $l_1, \dots, l_m$

Not true if L is not a field.

$$k \to k[x_1, \dots, x_n]$$

$\hookrightarrow$ Not a field.

$k \to L$ field ext. L fin. gen as alg / k

Thm $\Rightarrow$ L/k is finite ext.

Finite $\Rightarrow$ alg. ext^n

Any $l \in L$ satisfies (monic) irred.

$$l^n + a_{n-1} l^{n-1} + \cdots + a_0 = 0$$

$a_i \in k$.

k alg. closed.

$\Rightarrow$ all irred monics are

$$\boxed{(x-a)} = 0 \quad a \in k.$$

$k \xrightarrow{\sim} L \Rightarrow k \xrightarrow{\sim} L$

k ↳ alg cl. L ↳ algebraic

$$k \xrightarrow{\sim} k[X_1, \ldots, X_n]/m$$

k alg. closed.

i.e. mod m, every poly is eqv. to a constant!

$$x_1 \equiv a_1 \pmod{m}$$

$$\vdots$$

$$x_n \equiv a_n \pmod{m}$$

$$f(x_1, \ldots, x_n) \equiv f(a_1, \ldots, a_n) \pmod{m}$$

$$m = \{ f \mid f(a_1, \ldots, a_n) = 0 \}$$

$$= I(\{(a_1, \ldots, a_n)\})$$

$$= \langle x_1 - a_1, \ldots, x_n - a_n \rangle$$

$$\{\text{max. ideals}\} \underset{I}{\overset{V}{\longleftrightarrow}} \{\text{Points}\}$$

$$\begin{matrix} m & \longleftrightarrow & (a_1, \dots, a_n) \\ \shortparallel & & \\ \langle x_1 - a_1, \dots, x_n - a_n \rangle. & & \end{matrix}$$

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(5) If $V(I) = \phi$ then $I = (1)$.

i.e. $I \neq (1)$ then $V(I) \neq \phi$.

$$I \subset m \Rightarrow \underset{\{\text{point}\}}{\underset{\shortparallel}{V(m)}} \subset \underline{\underline{V(I)}}$$

max. ideals containing I	$\longleftrightarrow$	Pts of $V(I)$

(6) Consider $V(I) = X$.

If $f \in k[x_1, \dots, x_n]$ is zero on X.

i.e. $\{ f(x) = 0 \ \forall \ x \in X \}$

then $f^n \in I$ for some $n > 0$.

(0) $\left\{ \begin{array}{l} g = 0 \\ g \in I \end{array} \right\} + \left\{ f \neq 0 \right\}$

Trick: Extra "y" $k[x_1, \dots, x_n, y]$

(1) $\left\{ \begin{array}{l} g(x_1, \dots, x_n) = 0 \\ g \in I \end{array} \right.$ $\quad y f(x_1, \dots, x_n) - 1 = 0$

Sol^{ns} $(x_1, \ldots, x_n, y)$ of ①

$\Updownarrow \quad \downarrow \uparrow \quad y = 1/f(x_1, \ldots, x_n)$

$(x_1, \ldots, x_n)$ of ⓪

No sol^{n}s to ①.

① generates the unit ideal.

$k[x_1, \ldots, x_n, y]$

$$1 = \sum c_i(x_1, \ldots, y)\, g_i(x_1, \ldots, x_n) + c(x, y)\left(y f(x_1, \ldots, x_n) - 1\right)$$

Set $y = \dfrac{1}{f(x_1, \ldots, x_n)} \in k(x_1, \ldots, x_n)$

$$1 = \sum \boxed{c_i\left(x_1, \ldots, x_n, \frac{1}{f}\right)} g_i(x_1, \ldots, x_n)$$

holds in $\underline{\underline{k(x_1, \ldots, x_n)}}$

$$f^N = \sum (\text{Poly}) \cdot \boxed{\underline{g_i}}$$

$\in I$ in $k[x_1, \ldots, x_n]$

⑥

Nullst: $\underline{\underline{I \text{ radical}}}$

$$I \rightleftarrows V(I)$$

$$I(V(I)) \supset I$$

Nothing extra.

$f \in I(V(I))$ i.e.

$f \equiv 0$ on $V(I)$.

$\Rightarrow \underline{\underline{f^n \in I}} \Rightarrow f \in I$.

1. Georgina
2. Leyao
3. Dillon
4. Fabian
5. Reef
6. Samuel.