$\mathbb{A}^n_k$ and affine alg. sets $V(A)$ $A \subset k[x_1, \dots, x_n]$

$\hookrightarrow$ Topology – Zariski topology.

A topology on a set S is specified by
open sets or closed sets.

① ϕ, S are open
② Unions of opens are open
③ Finite intersections

① ϕ, S are closed
② Intersections
③ Finite unions.

Satisfy.

Zariski topology – Closed sets are $V(A)$
$A \subset k[x_1, \dots, x_n]$.

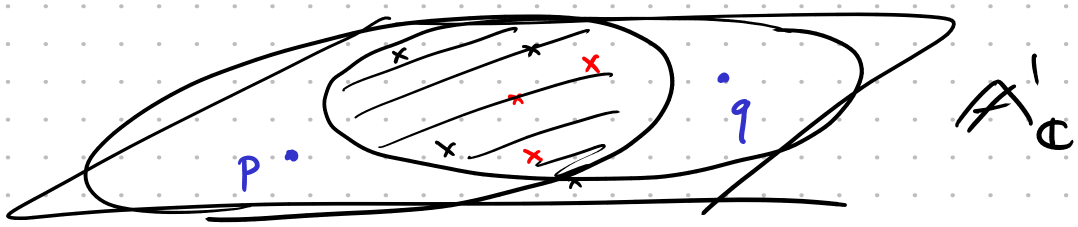
Ex. $k = \mathbb{C}$ (or $\mathbb{R}$) then $\mathbb{A}^n_{\mathbb{C}} = \mathbb{C}^n$ has the std. Euclidean top.

Every Zariski closed / open is closed / open in std top.

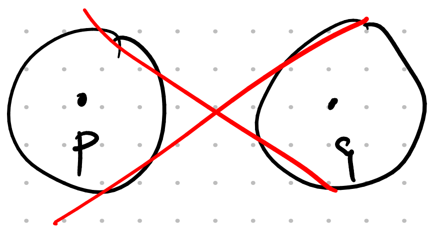
$n=1$: Zariski-closed subsets of $\mathbb{A}^1_k$.

$= \mathbb{A}^1_k$, or finite sets.

Open sets $= \phi, \mathbb{A}^1_k$, or complements of finite sets.



Non-Hausdorff!



Prop: $f \in k[x_1, \dots, x_n]$
$f : \mathbb{A}^n_k \to k = \mathbb{A}^1_k$ is continuous in Zariski topology.

The Nullstellensatz.

$$\text{Ideal of } k[x_1, \dots, x_n] \xrightarrow{V} \text{Subset of } \mathbb{A}^n_k$$

$$I(S) = \{ f \mid f \equiv 0 \text{ on } S \} \longleftarrow S$$

$\hookrightarrow$ An ideal.

Def: (Radical ideal) An ideal $I \subset R$ is radical if:
for every $f \in R$ such that $f^n \in I$ for some $n > 0$ we have $f \in I$.

Example: $(x) \subset k[x]$ is radical.
(i.e. if $f^n \in (x)$ i.e. $x \mid f^n$
then $f \in (x)$ i.e. $x \mid f$)

$(x^2) \subset k[x]$ is not radical.

$f = x \quad n = 2 \quad f^n \in I$ but $f \notin I$.

Equivalent - $I \subset R$ is radical iff R/I has no non-zero nilpotents.
$\hookrightarrow$ (an element f such that $\exists n > 0$ with $f^n = 0$.)

Why?
Take $f \in R$, consider $\bar{f} \in R/I$.

$\bar{f} = 0$ in R/I iff $f \in I$.
$\bar{f}^n = \overline{f^n} = 0$ in R/I iff $f^n \in I$.

$\bar{f}$ is nilpotent of $R/I \iff f^n \in I$ for some $n > 0$.

No non-zero nilp. in R/I: $\quad \bar{f}$ nilp $\iff \bar{f} = 0$
$f^n \in I$ for some $n > 0 \iff f \in I$.

Observe: Given $S \subset \mathbb{A}^n_k$.

Consider $I(S) = \{ f \in k[x_1, \dots, x_n] \mid f \equiv 0 \text{ on } S \}$

$I(S)$ is radical.

i.e., If $f^n \in I(S)$ for some $n > 0$ then $f \in I(S)$

$f^n \equiv 0$ on $S \Rightarrow f \equiv 0$ on S.

Algebra

$$\left\{ \begin{array}{c} \text{Radical} \\ \text{Ideals of} \\ k[x_1, \dots, x_n] \end{array} \right\} \underset{I}{\overset{V}{\rightleftarrows}} \left\{ \begin{array}{c} \text{Zariski closed} \\ \text{Subsets of} \\ \mathbb{A}^n_k \end{array} \right\}$$

Geometry

Thm (Nullstellensatz) If k is algebraically closed, then V & I are mutually inverse bijections. The resulting 1-1 corresp. is inclusion reversing.

$$I \subset J \Rightarrow V(I) \supset V(J)$$

$$I(S) \supset I(T) \Leftarrow S \subset T$$

Thm: Let $I \subset k[x_1, \dots, x_n]$. Then $V(I) = \emptyset$ iff $I = (1)$. (k alg. closed).

$I \leftarrow$ system of poly eq^{n}s.

$f = 0 \quad f \in I$

Then $1 \in I$.

$V(I) = \emptyset \Longrightarrow$

Suppose $I = \langle f_1, \ldots, f_m \rangle$

$$\left\{\begin{array}{l} f_1 = 0 \\ f_2 = 0 \\ \vdots \\ f_m = 0 \end{array}\right\} \quad V(I) = \phi \quad \Longrightarrow \quad 1 = g_1 f_1 + \cdots + g_m f_m$$

i.e. a "witness" to the non-existence!

Ex. $k = \mathbb{R}$ $\quad I = (x^2+1) \neq (1)$.

But $V(I) \subset \mathbb{A}^1_{\mathbb{R}}$ is empty!

$x^2+1=0$ has no sol^{n}s in $\mathbb{R}$.

but $1 \notin (x^2+1)$.

Thm: k alg. closed.

Then the maximal ideals of $k[x_1, \ldots, x_n]$ are

$$\langle x_1 - a_1, \ldots, x_n - a_n \rangle \text{ for some } (a_1, \ldots, a_n) \in \mathbb{A}^n_k$$

$$\| $$

$$I(\{(a_1, \ldots, a_n)\})$$

All max. ideals of $k[x_1, \ldots, x_n]$. $\longleftrightarrow$ Points of $\mathbb{A}^n_k$

max. ideal $= I(\{\text{point}\})$

Ex: $(x^2+1) \subset \mathbb{R}[x]$ is maximal but not of the form $\langle x-a \rangle$ for $a \in \mathbb{R}$.

$\rightarrow$ max id of $\mathbb{C}[x]$ are $\langle x-a \rangle$ for $a \in \mathbb{C}$.

on Wed / Thu

1. Verify the axioms of a topology for the Zariski top.

2. Proof of Nullstellensatz.