(1) In this problem, consider $A^k$ as the open subset of $P^k$ where the last homogeneous coordinate is non-zero.

The following maps from an open subset of $A^n$ to $A^m$ extend to regular maps from $P^n$ to $P^m$. Write down these extensions using homogeneous polynomials.

(a) $f: A^1 \rightarrow A^2$ defined by $f(t) = (t^2 - 1, t^3 - t)$.

(b) $f: A^2 \setminus V(xy) \rightarrow A^3$ defined by $f(x, y) = (x/y, y/x, 1/xy)$.

(2) Show that the natural map

$$\pi: A^2 \setminus \{(0,0)\} \rightarrow P^1$$

defined by $\pi(x, y) = [x : y]$ does not extend to a regular map $\pi: A^2 \rightarrow P^1$.

(3) (3-transitivity of $PGL_2$) Given three distinct points $p_1, p_2, p_3 \in P^1$, prove that there exists a unique projective linear transformation $P^1 \rightarrow P^1$ that sends

$0 = [0 : 1] \mapsto p_1, 1 = [1 : 1] \mapsto p_2, \text{ and } \infty = [1 : 0] \mapsto p_3$.

(4) (A cubic surface as a conic fibration) Suppose $\text{char } k \neq 2, 3$.

Let $S \subset P^3$ be the Fermat cubic surface

$$S = V(X^3 + Y^3 + Z^3 + W^3).$$

(a) Consider the linear projection $\pi: P^3 \rightarrow P^1$ defined by


Show that the center $L$ of the linear projection is contained in $S$.

(b) Show that $\pi: S \setminus L \rightarrow P^1$ extends to a regular map $\pi: S \rightarrow P^1$.

(c) What is the fiber of $\pi: S \rightarrow P^1$ over a point $[a : b] \in P^1$? (Be careful!)

(d) (Not to be turned in but highly recommended) Draw a (real) picture depicting $L, S$, a typical fiber of the linear projection $\pi: P^3 \setminus L \rightarrow P^1$, and a typical fiber of $\pi: S \rightarrow P^1$. 
