ALGEBRAIC GEOMETRY: WORKSHOP 10

The Grassmannian $\text{Gr}(2, 4)$

By definition, points of $\text{Gr}(2, 4)$ correspond to 2 dimensional subspaces $V \subset k^4$. By choosing a basis, we may represent $V$ as the column span of a $2 \times 4$ matrix $M$. Given $V$, the matrix $M$ is unique up to the action of $\text{GL}_2$ by right multiplication.

1. Consider the Plücker map $\text{Gr}(2, 4) \to \mathbb{P}^5 = \{[U_{12} : U_{13} : U_{14} : U_{23} : U_{24} : U_{34}]\}$.
   
   Let $W_{12} \subset \text{Gr}(2, 4)$ be the preimage of $\{U_{12} \neq 0\}$. Identify $W_{12}$ with $\mathbb{A}^4$ by making the first $2 \times 2$ block of $M$ equal to the identity. Identify $\{U_{12} \neq 0\}$ with $\mathbb{A}^5$ in the standard way, by making $U_{12} = 1$. Write down the Plücker map $W_{12} = \mathbb{A}^4 \to \mathbb{A}^5 = \{U_{12} \neq 0\}$.

   See that the image is a closed subset, and the map is an isomorphism onto it.

2. Show that the image of $\text{Gr}(2, 4) \subset \mathbb{P}^5$ is a degree 2 hypersurface.

3. Fix a “flag” in $k^4$, namely vector spaces $V_1 \subset V_2 \subset V_3$ of dimensions 1, 2, 3, respectively. By projectivising everything, we may view $\text{Gr}(2, 4)$ as the space of (projective) lines in $\mathbb{P}^3$. Then a flag corresponds to $p \in L \subset P$, where $p$ is a point, $L$ is a line, and $P$ is a plane in $\mathbb{P}^3$. Up to coordinate changes on $k^4$, all flags are equivalent, so you may take the standard (coordinate) flag for explicit calculations. Show that the following are closed subsets of $\text{Gr}(2, 4)$:
   (a) $\sigma_0 = \text{Gr}(2, 4)$,
   (b) $\sigma_1 = \{V \mid V \cap V_2 \neq 0\} = \{\text{Lines meeting } L\}$,
   (c) $\sigma_2 = \{V \mid V_1 \subset V\} = \{\text{Lines through } p\}$,
   (d) $\sigma_{11} = \{V \mid V \subset V_3\} = \{\text{Lines in } P\}$,
   (e) $\sigma_{21} = \{V \mid V_1 \subset V \subset V_3\} = \{\text{Lines in } P \text{ through } p\}$,
   (f) $\sigma_{22} = \{V = V_2\} = \{L\}$.

   One way to do this is to translate these conditions in terms of the matrix $M$.

4. See that these six sets correspond to six possible echelon forms of $M$.

5. Find the (co)-dimensions of these sets. Also see that
   
   $\sigma_2 \cong \mathbb{P}^2$, $\sigma_{11} \cong \mathbb{P}^2$, $\sigma_{21} \cong \mathbb{P}^1$.

6. Draw the poset formed by the sets $\sigma$ under inclusion.

   In general, $\text{Gr}(r, n)$ has a stratification by closed subsets indexed by Young tableaux that fit in an $r \times (n - r)$ box. These subsets are called “Schubert cells.” The inclusion relations correspond to the dominance order.