In class, we showed that a closed subset of $\mathbb{P}^n \times \mathbb{A}^m$ projects down to a closed subset of $\mathbb{A}^m$. The proof, however, did not give a good way of computing this closed subset. Today, we will see how to find the equations for this set for $n = 1$ and when $Z \subset \mathbb{P}^1 \times \mathbb{A}^m$ is defined by two polynomials.

The basic question is the following. Consider a system

$$F(X, Y) = 0 \text{ and } G(X, Y) = 0,$$

where $F$ and $G$ are homogeneous of degrees $d$ and $e$, respectively. When does the system have a non-zero solution $(X, Y)$?

1. Show that the system above has a non-zero solution if and only if $F$ and $G$ have at least one common linear factor.

2. Show that $F$ and $G$ have a common linear factor if and only if there exists a homogeneous polynomial $A$ of degree $e - 1$ and a homogeneous polynomial $B$ of degree $d - 1$ such that

$$AF + BG = 0.$$

3. Show that the existence of $A$ and $B$ as above is equivalent to the non-injectivity of the following linear map

$$m: k[X, Y]_{e-1} \oplus k[X, Y]_{d-1} \to k[X, Y]_{d+e-1}$$

defined by

$$m: (A, B) \mapsto AF + BG.$$

4. Let $F(X, Y, s, t) = sX^2 + tY^2$ and $G(X, Y, s, t) = X^2 + stXY + Y^2$. Using the previous part, write an equation in $s, t$ that is satisfied precisely when $F(X, Y, s, t)$ and $G(X, Y, s, t)$ have a common zero in $\mathbb{P}^1$. Your equation will have the form $\det \cdots = 0$. The matrix $\cdots$ is called the resultant matrix of $F$ and $G$ and its determinant is called the resultant.

5. The general case is similar. Given $F(X, Y, t_1, \ldots, t_m)$ and $G(X, Y, t_1, \ldots, t_m)$, the projection of $V(F, G) \subset \mathbb{P}^1 \times \mathbb{A}^m$ in $\mathbb{A}^m$ is cut out by the resultant of $F$ and $G$. 