ALGEBRAIC GEOMETRY: WORKSHOP 2

1. HOW TO DO THE LAST PROBLEM ON HW1 FOR ARBITRARY k?

For simplicity, let us take \( n = 2 \), and recall the proof for \( k = \mathbb{C} \). Let \( D \subset \mathbb{A}^{2 \times 2} \) be the set of diagonalizable matrices and \( B \subset \mathbb{A}^{2 \times 2} \) its complement. We show that neither \( D \) nor \( B \) are closed. To show that \( D \) is not closed, consider the family of matrices

\[
M_t = \begin{pmatrix} 1 & 1 \\ 0 & t \end{pmatrix}.
\]

See that \( M_t \in D \) for \( t \neq 1 \), but \( M_1 = \lim_{t \to 1} M_t \notin D \), which shows that \( D \) is not closed in the Euclidean topology, and hence also not in the Zariski topology. Similarly, by considering

\[
N_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},
\]

we see that \( B \) is not closed.

We mimic the same proof algebraically. Instead of limits, we use more basic topology (remember the Zariski topology is not Hausdorff!). The role of the family \( M_t \) parametrised by \( t \in \mathbb{R} \) is played by a similar family \( M_t \) parametrised by \( t \in \mathbb{A}^1 \).

Consider the map \( M: \mathbb{A}^1 \to \mathbb{A}^{2 \times 2} \) given by

\[
M: t \mapsto \begin{pmatrix} 1 & 1 \\ 0 & t \end{pmatrix}.
\]

Since \( M \) is defined by polynomial functions, it is continuous in the Zariski topology. Note that \( M^{-1}(D) = \mathbb{A}^1 \setminus \{1\} \) is not Zariski closed. Therefore, \( D \) is not Zariski closed. Similarly, we show that \( B \) is not Zariski closed.

2. MORE EXERCISES WITH IDEALS AND THEIR VANISHING LOCI.

(1) Let \( k \) be an algebraically closed field of characteristic not equal to 2. For \( c \in k \), let \( Z_c \) be the algebraic subset of \( \mathbb{A}^2_k \) defined by \( x^2 + y^2 = 1 \) and \( x = c \). Find \( I(Z_c) \) for all values of \( c \in k \) (Caution: Pay close attention to two special values of \( c \)).

(2) Draw a picture of the special and the general situation by taking \( k = \mathbb{R} \)

(3) What happens if the characteristic of \( k \) is 2?
3. **THE ZARISKI TOPOLOGY IS NOT HAUSSDORFF.**

Let $k$ be an algebraically closed field. Let us show that the Zariski topology on $\mathbb{A}^n_k$ is not Hausdorff. In fact, let us show that any two non-empty subset of $\mathbb{A}^n_k$ have a non-empty intersection.

(1) For $n = 1$, recall that the Zariski topology is the finite complement topology, and conclude.

(2) In general, show that every Zariski open $U \subset \mathbb{A}^n_k$ contains a *basic open*, namely an open set of the form

$$D(f) = \{ x \mid f(x) \neq 0 \}.$$

(3) Show that $D(f) \cap D(g) = D(fg)$, and conclude that any two non-empty opens must intersect.