Regular maps

Recall the notion of regular maps for quasi-affine varieties

\[ X \subset A^n, \quad Y \subset A^m \]

open subsets of Zariski closed sets.

Then a map \( f: X \to Y \) is regular if for every \( a \in X \) there exist \( f_1, \ldots, f_m \)

\[ g_1, \ldots, g_m \in k[x_1, \ldots, x_n], \quad g_i(a) \neq 0 \]

such that \( f = \left( \frac{f_1}{g_1}, \ldots, \frac{f_m}{g_m} \right) \)

in a neighbourhood of \( a \).

Using charts, we extend the definition to arbitrary algebraic varieties

\( f: X \to Y \) is regular if it is continuous and for every \( a \in X \) there exist (eqv. for every) charts \( (U, V, \phi) \) on \( X \) with \( a \in U \) & \( (U', V', \phi') \) on \( Y \) with \( f(a) \in V' \) such that the map \( \overline{f} \) below is regular.

\[
\begin{align*}
U \cap \phi(U') & \xrightarrow{f} U' \\
\downarrow & \quad \downarrow \\
\text{Open in } V & \xrightarrow{\overline{f}} V'
\end{align*}
\]
When $X$ and $Y$ are quasi-projective, there is a more user-friendly criterion.

Say $X \subseteq \mathbb{P}^n$, $Y \subseteq \mathbb{P}^m$.

**Prop.** $f : X \rightarrow Y$ is regular if and only if for every $a \in X$ there exist homog. poly $F_0, \ldots, F_m \in k[x_0, \ldots, x_n]$ such that not all $F_i$ are 0 at $a$ and $f = [F_0 : \ldots : F_m]$ in a neighborhood of $a$.

**Pf.** $(\Rightarrow)$ Suppose $f$ is regular.

Let $a = [a_0 : \ldots : a_n]$. wlog $a_n \neq 0$. Then $a$ lies in the affine chart $\{x_n \neq 0\} \sim \mathbb{A}^n$ of $\mathbb{P}^n$.

Let $b = f(a) = [b_0 : \ldots : b_m]$. wlog $b_m \neq 0$. Then $b$ lies in the affine chart $\{y_m \neq 0\} \sim \mathbb{A}^m$ of $\mathbb{P}^m$.

Restricting the charts to $X$ & $Y$ gives us charts...
\[ a \in X \land \exists X_n \neq 0 \quad \xrightarrow{T} \quad Y \land \exists Y_m \neq 0 \quad \xrightarrow{\overline{T}} \quad \xrightarrow{2} \quad \Lambda \supset U \quad \xrightarrow{\overline{T}} \quad V \subset \Lambda \]

Since \( \overline{T} \) is regular, there exist
\[ f_0, \ldots, f_{m-1}, g_0, \ldots, g_{m-1} \in \mathbb{R}[X_0, \ldots, X_{n-1}] \]

such that \( g_i(\overline{a}) \neq 0 \) for any \( i \) and
\[ \overline{T} = \left( \frac{f_0}{g_0}, \ldots, \frac{f_{m-1}}{g_{m-1}} \right) \quad \text{around} \quad \overline{a}. \]

Let us convert this back to homg. coordinates.
Set \( f_m = g_0 \cdots g_{m-1} \) and rename
\[ f_i \leftarrow \frac{f_i}{g_i}. \]

Then \( T \) is given around \( a \) by
\[ [X_0 : \cdots : X_m] \mapsto [f_0(X_0, \ldots, X_{m-1}) : \cdots : f_m(X_0, \ldots, X_{m-1})] \]

we are almost done. We just have to homogenise. Set \( \Delta = \max \deg f_i \) and
\[ F_i = X_m^\Delta f_i \left( \frac{X_0}{X_m}, \ldots, \frac{X_{m-1}}{X_m} \right) \]

Then \( T \) is given around \( a \) by
\[ [X_0 : \cdots : X_m] \mapsto [F_0(X_0, \ldots, X_m) : \cdots : F_m(X_0, \ldots, X_m)] \]
\((\Leftarrow)\) is even easier. Suppose we know that \(f\) has the stated form around \(a\). Let \(a = [a_0, \ldots, a_n]\) with \(a_n \neq 0\) & \(f(a) = [b_0, \ldots, b_m]\) with \(b_m \neq 0\).

Consider the restriction
\[X \cap \{x_n \neq 0\} \cap \{b_m \neq 0\} \rightarrow Y \cap \{y_m \neq 0\}\]

The std chart identifies LHS as a quasi-affine in \(\mathbb{A}^n\) & RHS as a quasi affine in \(\mathbb{A}^m\).

In terms of the charts, the map \(f\) looks like

\[f: (x_0, \ldots, x_{m-1}) \mapsto \left(\frac{F_0(x_0, \ldots, x_{m-1}, 1)}{F_m(x_0, \ldots, x_{m-1}, 1)}, \ldots, \frac{F_{m-1}(x_0, \ldots, x_{m-1}, 1)}{F_m(x_0, \ldots, x_{m-1}, 1)}\right)\]

which is regular. \(\square\)
Examples:

\[ f: \mathbb{P}^1 \rightarrow \mathbb{P}^2 \]

\[ f: [x:y] \mapsto [x^2:xy:y^2] \]

Image \( \subset \{ [u:v:w] \mid uw-v^2 \} \).

Inverse

\[ g: V(uw-v^2) \rightarrow \mathbb{P}^1 \]

\[ g: [u:v:w] \mapsto [u:v] \quad \{ \text{regular} \} \]

\[ [0:0:1] \mapsto [1:0]. \]

\[ g = [u:v:w] \mapsto [u:v] \text{ on } \{ W \neq 0 \} \]

\[ = [u:v:w] \mapsto [v:w] \text{ on } \{ U \neq 0 \}. \]

Geometry: What is the map

\[ [u:v:w] \mapsto [u:v] ? \]

\[ \mathbb{P}^2 \setminus \{ [0:0:1] \} \rightarrow \mathbb{P}^1 \]

\[ [U_0:V_0:W_0] \rightarrow [0:0:1] \]

"Linear projection"
\[ f : \mathbb{P}^1 \rightarrow \mathbb{P}^3 \]
\[ f : [x:y] \mapsto [x^3 : x^2y : xy^2 : y^3] \quad \text{regular!} \]

Image \( C \) \{ \{ [U_0 : U_1 : U_2 : U_3] \mid U_1^2 - U_0U_2, \quad U_2^2 - U_1U_3, \quad U_1U_2 - U_0U_3 \} = X \]

\[ g : X \rightarrow \mathbb{P}^1 \]
\[ g : [U_0 : U_1 : U_2 : U_3] \mapsto [U_0 : U_1] \quad \text{or} \quad [U_2 : U_3] \]

is an inverse!
Picture $g: \mathbb{P}^3 \to \mathbb{P}^1$

First $g: [u_0 : u_1 : u_2 : u_3] \mapsto [u_0 : u_1]$

$\mathbb{P}^3 \setminus V(u_0, u_1) \mapsto \mathbb{P}^1$

Copy $g' \mathbb{P}^1$

$[0:0:1:0]$ \hspace{2cm} $[0:0:0:1]$

$[u_0 : u_1 : u_2 : u_3]$ \hspace{2cm} $= V(u_2, u_3)$

$[u_0 : u_1]$ 

SO $g = \text{linear projection with "center of projection" } = V(u_0, u_1)$.

$g$ is not defined along the center of proj. but $g|_X$ extends to $X \cap \text{center of proj}$ to a regular map!
Generalisation

\[ f: \mathbb{P}^n \to \mathbb{P}^n \]
\[ [x:y] \mapsto [x:xy:\ldots:y] \]

is regular and maps \( \mathbb{P}^1 \) isomorphically onto
\[
\{ [u_0:\ldots:u_n] \mid \text{if } i\neq j \text{ then } u_iu_j - u_0u_k = 0 \}
\]

**Def.** The image of \( f \) is called the **rational normal curve** in \( \mathbb{P}^n \).

No reason to stop at curves

\[ v: \mathbb{P}^2 \to \mathbb{P}^5 \]
\[ v: [x:y:z] \mapsto [x^2:y^2:z^2:xy:yz:xz] \]
Then \( v \) is regular.

To find the image, it helps to label the homogeneous coordinates of \( \mathbb{P}^5 \) by
\[
\{ (ij,k) \mid i+j+k=2 ; i,j,k \geq 0 \}
\]
\[ \mathbb{P}^5 = \left\{ [U_{(2,0,0)} : U_{(0,2,0)} : U_{(0,0,2)} : \right. \\
\left. U_{(1,1,0)} : U_{(0,1,1)} : U_{(1,0,1)}] \right\} \]

Then the image lies in

\[ X = \bigvee \left( U_I U_J = U_k U_L \mid I+J = K+L \right) \]

**Thm:** \( V : \mathbb{P}^2 \to X \) is an isomorphism

**Pf (Sketch):**
- \( V \) is a bijection
- \( X \) is covered by the charts
  \[ \left\{ U_{(2,0,0)} \neq 0 \}, \quad \left\{ U_{(0,2,0)} \neq 0 \right\}, \]
  \[ \left\{ U_{(0,0,2)} \neq 0 \right\}. \]
- Invert is given by
  \[ [U_I] \mapsto [U_{(2,0,0)} : U_{(1,1,0)} : U_{(1,0,1)}] \]
on the first chart & likewise on the other two charts.

\[ \square \]
Def: $X \subset \mathbb{P}^5$ is called the Veronese surface.

$\nu : \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ is called the (2nd) Veronese embedding.

Why stop at a 2nd surface? Why stop at a surface?

Define $\nu_d : \mathbb{P}^n \rightarrow \mathbb{P}^N$ by

$[X_i] \mapsto [X^I] \quad I = (i_0, \ldots, i_n) \quad i_j \geq 0 \quad \sum i_j = d$

$N = \binom{n+d}{n} - 1$

Set $X = \bigvee \{ [U_i] \mid U_i U_j = U_k U_L \text{ when } I+J = K+L \}$

$C \subset \mathbb{P}^N$
Thm: \( V_d : \mathbb{P}^n \to X \) is an iso.

Pf: Similar to that of \( \mathbb{P}^2 \) (skipped).

\( V_d \) is called the \( d \)th Veronese embedding of \( \mathbb{P}^n \).

The existence of the Veronese embedding has the following consequence. Let

\[ F = \sum a_i X^i \]

be a homogenous poly of degree \( d \) in \( k[x_0, \ldots, x_n] \).

Consider \( V(F) \subset \mathbb{P}^n \)

Now consider \( \mathbb{P}^n \xrightarrow{V_d} \mathbb{P}^n \) by the \( d \)th Veronese. & consider the hyperplane
\[ H = V(\Sigma a_I \cup I) \subset \mathbb{P}^N. \]

**Note:** \[ V_d(V(F)) = V_d(\mathbb{P}^n) \cap H. \]

**Case:** \( \mathbb{P}^n \setminus V(F) \) is affine.

**Proof:** By the Veronese embedding

\[ V_d : \mathbb{P}^n \setminus V(F) \xrightarrow{\sim} (\mathbb{P}^n \setminus H) \cap V_d(\mathbb{P}^n) \]

\[ \mathbb{A}^1 \xrightarrow{\sim} \text{closed} \]

Isomorphism to a closed subspace of affine space! \( \square \)
Linear maps, projections, linear subspaces.

Suppose $M : k^n \rightarrow k^m$ is an injective linear map. Then we get a regular induced map

$$M : \mathbb{P}^n \rightarrow \mathbb{P}^m.$$ 

The image of $M$ is a linear subspace of $\mathbb{P}^m$, namely a set cut out by linear (homogeneous) equations.

In fact, the operation of taking the cone gives a bijection
Linear subspaces of $\mathbb{P}^n$ \iff (Nonzero) vector subspaces of $\mathbb{K}^{n+1}$

The smallest linear subspace containing a set $X \subseteq \mathbb{P}^n$ is called the linear span of $X$. If $\alpha \in \mathbb{K}^{n+1}$ is any non-zero point on the line represented by $x \in X$, then the linear span of $X$ corresponds (under the bijection above) to the vector space span of $\{ \alpha | x \in X \} \subseteq \mathbb{K}^{n+1}$.

Now suppose $M : \mathbb{K}^{n+1} \to \mathbb{K}^{m+1}$ has a nonzero kernel $K \subseteq \mathbb{K}^{n+1}$. Then the map $M : [x] \mapsto [Mx]$ is regular on $\mathbb{P}^n \setminus \mathbb{P}K$.

If $M$ is surjective, then $M$ is called the linear projection of $\mathbb{P}^n$ onto $\mathbb{P}^m$ with center $\mathbb{P}K$. 