Regular functions and regular maps

\( k = \text{Alg. closed field} \)

Recall from last time:

- \( X \subset \mathbb{A}^n_k \) affine algebraic set.
- \( f : X \to k \) regular if it is the restriction of a polynomial function.

\[
k[X] = \text{\( k \)-algebra of regular functions on } X
\]
\[
\cong k[x_1, \ldots, x_n] / I(X).
\]
\[
= \text{Finitely generated nilpotent free } \ k\text{-algebra.}
\]

Observe - Any finitely generated nilpotent free \( k \)-algebra is of the form \( k[X] \) for some \( X \).

Why? Let \( A \) be such an algebra.

Let \( a_1, \ldots, a_n \in A \) be a set of generators.

Then we have a map

\[
\phi : k[x_1, \ldots, x_n] \to A
\]
\[
a_i \mapsto a_i.
\]

This map is surjective because \( \{a_i\} \) generates \( A \). By the first iso thm

\[
A \cong k[x_1, \ldots, x_n] / I
\]
where \( I = \ker \varphi. \)

Since \( A \) is nilpotent free, \( I \) is radical.

Then take \( X = V(I) \).

By the Nullstellensatz,
\[
\begin{align*}
\kappa[x] &= \kappa[x_1, \ldots, x_n] / I(x) \\
&= \kappa[x_1, \ldots, x_n] / I \\
&\cong A
\end{align*}
\]

\( \qed \)

As a result we have the dictionary.

\underline{Algebra}  \hspace{2cm} \underline{Geometry}

- Finitely generated reduced \( k \)-alg. \( A \)
  - Alg of regular functions on affine alg set \( X \).
- Max ideal \( \mathfrak{m} \) \( A \)
  - Point of \( X \).
- Given \( J \subset A \)
  - Given \( J \subset \kappa[x] \)
  \[
  V(J) = \{ m \mid m \supseteq J \}.
  \]
  \[
  V(J) = \{ x \mid f(x) = 0 \} \quad \forall f \in J \}
  \]

In particular \( V(J) = \emptyset \) if \( J = (0) \).
Regular Maps

\( X \subset \mathbb{A}^n, Y \subset \mathbb{A}^m \) affine algebraic sets.
\( f : X \to Y \) is a regular function if

\[ \exists f_1, \ldots, f_m \in \mathbb{k}[X] \quad \text{such that} \]

\[ f(x) = (f_1(x), \ldots, f_m(x)) \quad \forall x \in X. \]

Equivalently, if there exist \( F_1, \ldots, F_m \)
in \( \mathbb{k}[X_1, \ldots, X_n] \) such that

\[ f(x) = (F_1(x), \ldots, F_m(x)) \quad \forall x \in X. \]

**Ex 1:** \( f : X \to \mathbb{A}^1 \) regular map

\( \iff \) \( f \) is a regular function.

**Ex 2:** \( L : \mathbb{A}^n \to \mathbb{A}^m \) linear transform

is regular.

**Ex 3:** Projections \( \mathbb{A}^n \to \mathbb{A}^1 \)

**Ex 4:** Compositions of regular maps
are regular.
Ex 5: \( X \subseteq A^n \) Zariski closed. The inclusion \( X \rightarrow A^n \) is regular.

Def: A regular \( f : X \rightarrow Y \) is an isomorphism if there exists a regular inverse map \( j : Y \rightarrow X \).

Ex 6: \( X = A^1 \)
\( Y = \{ y^2 - x^3 = 0 \} \subseteq A^2 \)

\( f : X \rightarrow Y \)
\( t \mapsto (t^2, t^3) \) is a regular bijection but not an isomorphism!
How does one see that it's not an iso? Wait and see....

Let \( \varphi : X \rightarrow Y \) be any map.
Then we get an induced map

\( \varphi^* : \text{Functions on } Y \rightarrow \text{Functions on } X \)
\( f \mapsto f \circ \varphi \).
Proposition: \( \varphi \) is regular if and only if \( \varphi^* \) sends regular functions on \( Y \) to regular functions on \( X \).

Proof: Suppose \( \varphi \) is regular. If \( f: Y \to A^1 \) is a regular function then \( \varphi \circ f \) is regular because composition of regular maps is regular.

Conversely, suppose \( \varphi^*(f) \) is regular for every regular \( f \). Let \( \varphi(x) = (\varphi_1(x), \ldots, \varphi_m(x)) \) we want to show each \( \varphi_i(x) \) is regular. But \( \varphi_i = \varphi^*(x_i) \) and \( x_i \in k[Y] \) is regular.

Thus a regular map \( \varphi: Y \to X \) induces a \( k \)-alg. hom \( \varphi^*: k[Y] \to k[X] \).

Proof: Let \( \alpha: k[Y] \to k[X] \) be a \( k \)-alg. hom. Then there is a unique regular \( \varphi: X \to Y \) such that \( \alpha = \varphi^* \).

Proof: Suppose \( Y = V(\mathcal{F}) \subset A^m \) and \( X = V(\mathcal{I}) \subset A^n \).
Then \( k[Y] = k[y_1, \ldots, y_m] / J \)
\( k[X] = k[x_1, \ldots, x_m] / I. \)

Let \( \phi_i = \alpha(y_i) \in k[x] \)

Consider \( \phi := (\phi_1, \ldots, \phi_m) : X \rightarrow \mathbb{A}^m. \)

Let us check that \( \phi \) maps \( X \) to \( Y. \)

To see this, we must show that
\[
    f(\phi_i(x), \ldots, \phi_m(x)) = 0 \quad \forall \ x \in X, \quad f \in J.
\]

But
\[
    f(\phi_1(x), \ldots, \phi_m(x)) \\
    = f(\alpha(y_1), \ldots, \alpha(y_m)) \\
    = \alpha(f(y_1, \ldots, y_m)) \\
    = \alpha(0) = 0.
\]

So \( \phi : X \rightarrow Y. \) Note \( \phi^*(y_i) = \alpha(y_i) \)

so \( \phi^* = \alpha \) because \( y_i \) generate \( k[Y]. \)

Finally, it \( \phi : X \rightarrow Y \) is such that
\( \phi^* = \alpha \), and \( \phi = (\phi_1, \ldots, \phi_m) \), then
\( \phi^*(y_i) = \phi_i = \alpha(y_i) \), so there is only one possible \( \phi. \)

\( \square. \)
Conseq: $X \rightarrow k[X]$ defines an equivalence of categories

\[
\begin{align*}
\{ & \text{Affine alg} \quad ? \quad \} \quad \longrightarrow \quad \{ & \text{Fin gen reduced} \\
& k\text{-algebras} \quad \text{with } k\text{-alg.} \\
& \text{homs} \quad \} \\
\{ & \text{sets with} \\
& \text{regular maps} \quad \} \\
\end{align*}
\]

Ex:

\[
X = \mathbb{A}^1 \\
y = V(y^2 - x^3) \subset \mathbb{A}^2
\]

\[
k[X] = k[t] \\
k[Y] = k[x] / (y^2 - x^3)
\]

\[
\varphi : X \rightarrow Y \\
\varphi(t) = (t^2, t^3)
\]

\[
\varphi^* : k[Y] \rightarrow k[X] \\
x \mapsto t^2 \\
y \mapsto t^3.
\]

\[\varphi^* \text{ is not an isomorphism!}
\]

Any element in the image of $\varphi^*$ has vanishing linear term.
Def: Affine algebraic variety
   = Affine algebraic set.

We eventually want to define more general algebraic varieties. The first step is

Def: Quasi-affine varieties = Zariski open subsets of affine alg. var.

We now define regular functions and regular maps for quasi-affines.

Def: $U \subset X$ open.
    $f: U \rightarrow k$ regular if the following holds - for $x \in U$ there exists an open $U_x$ containing $x$ & $F_x, G_x \in k[x]$ such that $G_x$ is nowhere 0 on $U_x$ and
    $f = F_x / G_x$ on $U_x$.

Example: $U = \mathbb{A}^1 - \{0\} \subset \mathbb{A}^1$.
Then $\frac{1}{t}$ is regular on $U$. 
2) \[ X = \{ xy - z^2 = 0 \} \subseteq \mathbb{A}^3 \]
\[ U = X - \{ (x,0,0) \mid x \in k \} \]
\[ f = \frac{x}{z} \text{ or } \frac{z}{y} \text{ is regular on } U. \]

Before we proceed, we must show that we get the same notion of regular as before for affines.

\[ \text{Prop: Let } X \subset \mathbb{A}^n \text{ be Zar. closed.} \]
\[ f: X \rightarrow k \text{ is regular in the new sense (locally poly/poly)} \iff \text{ it is regular in the old sense (globally a polynomial).} \]

\[ \text{Pf: Let } x \in X. \text{ There exist } U_x, F_x, G_x \]
\[ \text{such that } f = \frac{F_x}{G_x} \text{ on } U_x \text{ and } x \in U_x. \]
\[ \text{Say } U_x = X - V(I_x). \text{ Take } H_x \in I_x \]
\[ \text{such that } H_x(x) \neq 0. \text{ Replace } U_x \]
\[ \text{by } U'_x = X - V(H_x) \subseteq U_x. \]
\[ F_x \text{ by } A_x = F_x H_x \quad \text{and} \]
\[ G_x \text{ by } B_x = G_x H_x. \]
Then \( f = \frac{A_x}{B_x} \) on \( U'_x \),

ac \( U'_x \) and \( A_x, B_x = 0 \) on the complement of \( U'_x \).

Now \( \{ B_x \} \) have no common zero, so by the Nullstellensatz, they generate the unit ideal \( \mathfrak{a} \) of \( k[X] \). Write

\[ 1 = C_1 B_{x_1} + \ldots + C_n B_{x_n} \]

where \( C_i \in k[X] \).

Multiply both sides by \( f \)

\[ f = \sum C_i B_{x_i} \cdot f \]

and note

\[ B_{x_i} f = A_{x_i} \text{ on } X \]

so

\[ f = \sum C_i A_{x_i} \in k[X] \]

\( \square \).

Having defined regular functions, we can define regular maps just as before.

Ref: \( U \subseteq \mathbb{A}^n \), \( V \subseteq \mathbb{A}^m \), opens in closed.

\( \varphi : U \rightarrow V \) regular map

\( \varphi = (\varphi_1, \ldots, \varphi_m) \) where \( \varphi_i \) is reg. fun.
Obs: ① Pull backs of reg. fun under reg maps are regular
② Compositions of reg. fun are regular

Example (Important).

\[ X = \mathbb{A}^1 - \{0\} \]
\[ Y = \mathbb{V}(xy - 1) \subset \mathbb{A}^2. \]

\[ \varphi : Y \rightarrow X \]
\[ (x, y) \mapsto x. \]

\[ \psi : X \rightarrow Y \]
\[ x \mapsto (x, \frac{1}{x}) \]

Regular

\[ \varphi \circ \psi = \text{id}, \quad \psi \circ \varphi = \text{id}. \]
So \( X \cong Y. \)

That is the quasi-affine \( X \) is actually affine!

Ring of reg. fun on \( X = \)
\[ k[x, y]/(xy - 1) \cong k[t, t'] / (tk(t)) \]

by \( x \mapsto t, y \mapsto t'. \)
Example (Important)

\[ X = \bigcup_{i=1}^{n} V(f_i), \]
\[ Y = V(y f-1) \subset \bigcup_{i=1}^{n+1} \]

\[ \phi : Y \to X \quad \text{regular} \]
\[ (x, y) \mapsto x \]

\[ \psi : X \to Y \quad \text{regular} \]
\[ x \mapsto (x, \frac{1}{f(x)}) \]

\[ \phi \circ \psi = \text{id} \quad \psi \circ \phi = \text{id}. \]

So \[ X \cong Y. \]

hence

\[ k[X] = \frac{k[x_1, \ldots, x_n, y]}{(f(x)y-1)} \]

\[ \cong \left \{ \frac{p}{f^m} \mid p \in k[x_1, \ldots, x_n], m \geq 0 \right \} \]

\[ \subset k(x_1, \ldots, x_n, y) \]

by the map \[ x_i \mapsto x_i, \quad y \mapsto \frac{1}{f}. \]
A non-affine variety

\[ X = \mathbb{A}^2 \setminus \{0\} \]

We have a map
\[ k[x] \to k(x,y) \]
\[ f \mapsto \frac{f}{G} \]
where \( f = E \) on some open \( U \) in \( X \).

The choice of \( U \) does not matter —
First any two opens in \( X \) intersect
\( \Rightarrow \) any open is dense.
So if \( f = \frac{F_1}{G_1} \) on \( U_1 \)

\[ = \frac{F_2}{G_2} \] on \( U_2 \)

then \( G_2 F_1 - F_1 G_2 = 0 \) on \( U_1 \cap U_2 \)

\[ = 0 \] on \( \mathbb{A}^2 \)
by continuity. So \( F_1 / G_1 = F_2 / G_2 \) in \( k(x,y) \).

Write \( X = \mathbb{A}^2 - V(x) \cup \mathbb{A}^2 - V(y) \)
Now the reg. func. on \( \mathbb{A}^2 - V(x) \) in \( k(x,y) \) are \( \frac{1}{x^a} \frac{1}{y^b} \)
Similarly reg. func. on \( \mathbb{A}^2 - V(y) \) are
\{ \frac{f}{y^m} \}.

A reg fun on $X$ must lie in the intersection but the intersection

$$\left\{ \frac{f}{x^n} \big| f \in \mathbb{k}[x,y] \right\} \cap \left\{ \frac{f}{y^m} \big| f \in \mathbb{k}[x,y] \right\}$$

$$\subset \mathbb{k}[x,y].$$

So $\mathbb{k}[X] = \mathbb{k}[x,y] = \mathbb{k}[x^2].$

To conclude that $X$ is not affine, see that the ideal $(x,y) \subset \mathbb{k}[X]$ is non unit but $V(x,y) = \emptyset$ in $X$. This does not happen for affine $X$.

\[\square\]