You should be able to do all the questions on all the homeworks and all the workshop questions. In addition, here are more practice questions.

1. Prove that \( \mathbb{P}^1 \times \mathbb{P}^1 \) and \( \mathbb{P}^2 \) are birational but not isomorphic.
2. Prove that \( \mathbb{P}^1 \times \mathbb{P}^3 \) and \( \mathbb{P}^2 \times \mathbb{P}^2 \) are birational but not isomorphic.
3. Let \( X = V(xy, yz) \subset \mathbb{A}^3 \). Decompose \( X \) into irreducible components. What are their dimensions?
4. Let \( X \) be an irreducible quasi-projective variety. State the definition of the dimension of \( X \). Show that \( X \) has subvarieties of every dimension from 0 to \( \dim X \).
5. Prove that a closed subset \( X \subset \mathbb{A}^n \) is irreducible if and only if the ideal \( I(X) \subset k[x_1, \ldots, x_n] \) is prime.
6. Show that any birational automorphism of \( \mathbb{P}^1 \) is a projective linear transformation.
7. For every \( n \geq 2 \), give an example of a birational isomorphism \( \mathbb{P}^n \to \mathbb{P}^n \) that does not extend to a regular map.
8. Let \( M \) be an \( n \times n \) matrix. A vector \( v \in k^n \) is called a generator for \( M \) if the set \( v, Mv, M^2v, \ldots, M^{n-1}v \) spans \( k^n \). For example, if \( M \) is the diagonal matrix with distinct entries \( a_1, \ldots, a_n \), then the vector \( [1, \ldots, 1] \) is a generator. Let \( U \subset \mathbb{P} \text{Mat}_{n \times n} \) be the set (up to scaling) of matrices that admit a generator. Prove that \( U \) is Zariski open.

(Hint: Consider the space of pairs \((M, v)\) such that \( v \) is not a generator of \( M \).)
9. Let \( \phi: \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^{(n+1)(b+1)-1} \) be the composite \( \tau = \sigma \circ (\nu_a \times \nu_b) \), where the map \( \nu_m: \mathbb{P}^1 \to \mathbb{P}^m \) is the degree \( m \) Veronese and \( \sigma: \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{(m+1)(n+1)-1} \) is the Segre map.
   (a) Write the map \( \tau \) explicitly: where does \( ([X : Y], [U : V]) \) go?
   (b) Let \( F \subset k[X, Y, U, V] \) be bihomogeneous of bidegree \( a, b \) with \( a, b > 0 \). Show that \( \mathbb{P}^1 \times \mathbb{P}^1 \setminus \mathbb{V}(F) \) is affine.
10. Find all singular points on the curve \( V(X^3Y - Z^4) \subset \mathbb{P}^2 \).
11. Let \( X \subset \mathbb{P}^n \) be a closed subvariety. Fix positive integers \( r \) and \( l \). Let \( \Sigma \subset \text{Gr}(r, n+1) \) be the set of \( \Lambda \) such that \( \dim(\mathbb{P}\Lambda \cap X) \geq l \). Prove that \( \Sigma \) is a closed subvariety.
12. Prove that not all degree 4 hypersurfaces in \( \mathbb{P}^3 \) contain a line.