Generic sections and special divisors.

(1) Let \((V, D)\) be a base-point free linear series on a compact Riemann surface \(X\). Show that there exists \(\sigma \in V\) such that \((\sigma)\) is multiplicity free. (This is a special case of something called Bertini’s theorem.)

(2) Let \(D\) be a divisor and \(E\) an effective divisor. Show by induction on \(\deg E\) that

\[
h^0(D - E) \geq \max(0, h^0(D) - \deg E).
\]

Also show that the inequality is sharp—that is, given a \(D\), there exists an \(E\) of every degree such that equality holds.

(3) Let \(D\) be a divisor on \(X\) of degree \(d\). Show that we have

\[
h^0(D) \begin{cases} 
  = d - g + 1 & \text{if } d > 2g - 2 \\
  \geq d - g + 1 & \text{if } 2g - 2 \geq d \geq g \\
  \geq 0 & \text{if } g - 1 \geq d \geq 0.
\end{cases}
\]

Also show that the inequalities are sharp—that is, there exist \(D\) of every degree where equalities hold.

*Hint:* Write \(D = H - E\), where \(H\) and \(E\) are effective and \(\deg H\) is huge.

*Remark:* A divisor (class) \(D\) for which \(h^0(D)\) is strictly larger than the bounds above is called *special*. Much of the study of algebraic curves (and their moduli space) involves understanding special divisors on curves.

Quadric surfaces and genus 4 curves.

“Quadric” is a commonly used short-form for “degree 2.”

(4) Show that an irreducible quadric hypersurface in \(\mathbb{P}^3\) is isomorphic to either

\[
X^2 + Y^2 + Z^2 + W^2 = 0
\]

or

\[
X^2 + Y^2 + Z^2 = 0.
\]

(5) Show that a smooth quadric hypersurface in \(\mathbb{P}^3\) is isomorphic to \(\mathbb{P}^1 \times \mathbb{P}^1\).

(6) Recall that a line through two points \(P\) and \(Q\) in \(\mathbb{P}^n\) is given parametrically by

\[
L = \{up + vQ \mid [u:v] \in \mathbb{P}^1\}.
\]

Use this to describe all the lines on the smooth quadric \(X^2 + Y^2 + Z^2 + W^2 = 0\) and the singular quadric \(X^2 + Y^2 + Z^2 = 0\).

Let \(X\) be a compact Riemann surface of genus 4. Suppose \(X\) is not hyperelliptic. Then \(X\) is embedded in \(\mathbb{P}^3\) by the canonical linear series. In the following problems,
use that $X \subset \mathbb{P}^3$ is the intersection of an (irreducible) quadric hypersurface and a cubic hypersurface.

(7) Using geometric Riemann–Roch and the geometry of quadric hypersurfaces from the previous problems, show that there exist exactly two $g^1_3$’s on $X$ if $Q$ is smooth, and exactly one $g^1_3$ on $X$ if $Q$ is singular.

(8) Suppose $X$ is a compact Riemann surface of genus 4 with two $g^1_3$’s, say $D_1$ and $D_2$. Use Riemann–Roch to show that

$$D_1 + D_2 \sim K_X.$$  

Similarly, if $X$ has only one $g^1_3$, say $D$, then show that

$$2D \sim K_X.$$  

**Branched covers and monodromy.**

(g) Let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree $d$, defined by $F(X, Y, Z) = 0$. Assume that $[0 : 0 : 1]$ does not lie on $X$. Consider the projection $C \to \mathbb{P}^1$ given by $[X : Y : Z] \mapsto [X : Y]$. Show that the ramification divisor of $C$ is the zero locus of on $C$ of the homogeneous polynomial $\frac{\partial F}{\partial Z}$. Using Riemann–Hurwitz, conclude that the genus of $C$ is $d(d - 1)/2$.

(10) Let $C$ be the Fermat curve

$$X^d + Y^d + Z^d = 0.$$  

Consider the projection $\phi: C \to \mathbb{P}^1$ that drops the $Z$ coordinate (see (g)). Find $\text{br} \phi \subset \mathbb{P}^1$ and determine the monodromy map

$$\pi_1(\mathbb{P}^1 \setminus \text{br} \phi) \to S_d.$$  

(11) Let $X$ be a compact Riemann surface of genus $g$. Given a finite subset $B \subset X$ of even cardinality, show that there are $2^{2g}$ double covers of $X$ with branch divisor $B$ (If $B$ is empty, then one of them will be disconnected).