Linear series.

$X$ a compact Riemann surface.
$D$ a divisor on $X$.

$H^0(X, \mathcal{O}(D))$ is a fin dim $\mathbb{C}$-vector space. In fact, let $D$ be effective,
then
$$\dim H^0(X, \mathcal{O}(D)) \leq \deg D + 1.$$ (In general we get $\leq \deg(D^+) + 1$)

$$H^0(X, \mathcal{O}(D)) = \{ f \in \mathcal{M}_X \mid (f) + D \geq 0 \} = \{ \text{Hol. sec. } \subseteq \text{corresponding } \mathcal{L} \}.$$

$\mathcal{P}H^0(X, \mathcal{O}(D)) = \{ E \mid E \sim D \& E \geq 0 \} = |D|$

Def: A linear system is a pair $(V, D)$ where $V \subset H^0(X, \mathcal{O}(D))$. is a vector subspace.

- Base point.

Ex: $X \longrightarrow \mathbb{P}^n$. \[ L = \varphi^* \mathcal{O}(1) \]
\[ D = \varphi^* \mathcal{H} \]

Get a linear series $V \subset H^0(X, L)$ given by $V = \{ \varphi^*(\Sigma a_i x_i) \}$

$|V| \subset |D|$ \quad \text{base-point free}

$\exists$ Restrictions of hyperplanes?.

$\deg (V, D) := \deg D$ \quad A bpf lin sys $\varphi$
$\dim (V, D) := \dim V$ \quad $\dim (n+1)$ gives

$i : X \rightarrow \mathbb{P}^n.$
When is $i$ an embedding?

What is an embedding? $X, Y$ complex manifolds. $\varphi: X \to Y$ is an embedding if

1. $X \to \varphi(X)$ is a homeomorphism.
2. $\forall p \in X$ and $f \in O_{X, p}$ $\exists g \in O_{Y, \varphi(p)}$ such that $f = g|_{X}$.

Restatement of (2): The restriction map $O_{Y} \to f_{*}O_{X}$ is a surjection.

**Non-Example** $\Delta = \text{disk } \{ z \in \mathbb{C} \mid |z| < 1 \}$

$\varphi: \Delta \to \mathbb{C}^{2}$

$z \mapsto (z, z^{3})$.

Then $\varphi$ satisfies (1), but not (2).

**Example** $z \mapsto (z^{2}, z^{3})$.

For a Riemann surface $X$, for (2), it suffices to have $g \in O_{Y, \varphi(p)}$ st. $g|_{X}$ is a uniformizer at $p$.

Let us examine (2) for $X \to \mathbb{P}^{n}$ given by $(V, L)$. Let $p \in X$.

$\varphi(p) = [\sigma_{0}(p) : \cdots : \sigma_{n}(p)]$.

By a linear change of coordinates $\sigma_{0}(x) \neq 0$ & $\sigma_{1}(p), \cdots, \sigma_{n}(p) = 0$.

So locally near $p$:

$\varphi(x) = (\frac{\sigma_{1}}{\sigma_{0}}(x), \cdots, \frac{\sigma_{n}}{\sigma_{0}}(x)) \in \mathbb{A}^{n}$.
\( O_{\mathbb{A}^n,0} = \text{(conv) power series} \).

\( \exists \ f \in O_{\mathbb{A}^n,0} \text{ s.t. } f/x \text{ is a uniformizer at } p \)
\( \iff \exists \ i \text{ s.t. } \frac{\partial_i f}{\partial_i x} \text{ is a uniformizer at } p \)
\( \iff \exists \ i \text{ s.t. } \sigma_i \text{ has a simple zero at } p \)
\( \iff \exists \ \sigma \in V \text{ that has a simple zero at } p. \)

**Def.** \((V,D)\) a linear system. We say it **separates** tangent vectors at \(p\) if \(\forall \sigma \in V \text{ s.t. } \sigma(p) = 0 \text{ and } \sigma(q) \neq 0\).

Examine (i): \(X\) compact.
\[ X \to \mathbb{P}^n \text{ homeo onto image iff one-one.} \]
If \(\forall p \neq q, \exists \sigma \in V \text{ s.t. } \sigma(p) = 0 \text{ and } \sigma(q) \neq 0\).

**Def.** \((V,D)\) separates points if \(\cdots\).

**Thm.** Let \((V,D)\) be a base point free linear system.
Then the induced map \(\psi: X \to \mathbb{P}^n\) is an embedding iff it separates points and tangent vectors.

**Explain the terminology.**

**Extended Example:** \(X = \text{ compactification of } \mathbb{A}^3\) with \(Z = x^3 + y^2 + 1, W = y + y^4 + 1\).
Let us write down \(H^0(X, O(D))\).
\(\langle 1, x, Z \rangle\), a linear system.
\[ (1) = 3 \infty \]
\[ (\infty) = s + t + \infty \]
\[ (z) = p + q + r. \]

Bpt. Also separates points and tangent vectors.

\[ \varphi : X \rightarrow |P^2| [A:B:C] \]

Note: \( \varphi(x) \) satisfies \( AC^2 = A^3 + B^3 \).

Genus of a plane curve of degree \( d \) is

\[ \frac{(d-1)(d-2)}{2}. \]

(See next page).

**Thm:** (Riemann-Roch)

\( X \) a compact R.S. \( K \) a canonical divisor.

\( D \) a divisor on \( X \) of degree \( d \)

\[ \dim H^0(X, O(D)) = d - g + 1 + H^0(X, O(X-D)) \]
What is $g(C)$?

$C \rightarrow \mathbb{P}^1$, $L = \mathcal{O}(1)$, $X, Y$.

Want $X, Y$ not to vanish simultaneously on $C$ (i.e. $[0:0:1] \not\in C$).

$\mathbb{P}^2 - \{[0:0:1]\} \rightarrow \mathbb{P}^1$

$[x:y:z] \rightarrow [x:y]$.

Q: What is $\text{Ram}(\varphi)$?

$C \ni p = [A:B:c]$ assume $B \neq 0$

$= \begin{bmatrix} A \\ B \\ 1 \end{bmatrix} \begin{bmatrix} c \\ e \end{bmatrix} \epsilon C^2_{(x,z)}$

$C \cap C^2$ def. by $f(x, z) = 0$ $f(x, z) = F(x, 1, z)$.

$p$ is ramified iff $\frac{\partial f}{\partial z}(p) = 0$.

More: $\text{Ord}_p(\text{Ram} \varphi) = \text{Ord}_p \frac{\partial f}{\partial z} = \text{Ord}_p \left( \frac{\partial F}{\partial z} \right)$. 
So \( \text{Ram}(\varphi) = \text{div} \left( \frac{\partial F}{\partial z} \right) \).

(section 8, \( O(d-1) \)).

\[ \deg \text{Ram}(\varphi) = d \cdot (d-1) \]

so \( 2g_c - 2 = (-2) \cdot d + d \cdot (d-1) \)

\[ = d^2 - 3d \]

\[ g_c = \frac{d^2 - 3d + 2}{2} \]

\[ g_c = \frac{(d-1)(d-2)}{2} \]

Q.1) Are all compact R.S. of genus \( \frac{(d-1)(d-2)}{2} \)
plane curves of degree \( d \) ?

2) What about R.S. of genus not of this form ?

3) What's the "Simplest" way to exhibit a R.S. as a projective curve ?