Claire Voisin on the question of rationality

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Warm-up

Can you recognise these numbers?

\((\frac{3}{5}, \frac{4}{5}), (\frac{5}{13}, \frac{12}{13}), (\frac{8}{17}, \frac{15}{17}), (\frac{7}{25}, \frac{24}{25}), (\frac{20}{29}, \frac{21}{29}), \ldots\) 

These are solutions \((x, y)\) of

\[x^2 + y^2 = 1.\]

All the solutions:

\[x = \frac{1 - t^2}{1 + t^2}, \quad y = \frac{2t}{1 + t^2}.\]
Two systems of equations . . .

Variables: $x, y$
Equations: $x^2 + y^2 = 1$.  

. . . are equivalent by

$x = \frac{1-t^2}{1+t^2}, \quad y = \frac{2t}{1+t^2}$

$x, y \leftarrow \rightarrow t$

$\frac{y+1-x}{y+1+x}$

$x^2 + y^2 = 1$
An algebraic variety is the set of solutions of a system of polynomial equations.

**Examples**

\[ x^2 + y^2 = 1 \]

\[ x^3 + y^3 + z^3 + 1 = (x + y + z + 1)^3 \]

A Kummer Surface
An algebraic variety is the set of solutions of a system of polynomial equations.

**Example (The best one)**

\[ \mathbb{A}^n = \text{The ambient space (no equations)!} \]
A variety $X$ is rational if it is birational to $\mathbb{A}^n$. 

System of equations $\leftarrow$ Coördinate change $\rightarrow$ No equations!
Rational varieties

Which varieties are rational?

1. The variety defined by $x^2 + y^2 = 1$ is rational.
2. Varieties defined by linear equations are rational.
3. Varieties defined by one quadratic equation are rational (over $\mathbb{C}$).
4. Varieties defined by one cubic?
   4.1 Cubic curves: not rational (ancient)
   4.2 Cubic surfaces: rational (Castelnuovo, Enriques: Early 1900s)
   4.3 Cubic threefolds: not rational (Clemens–Griffiths: 1972)
   4.4 Cubic fourfolds and higher: ???
Artin–Mumford (1971): If \( X \) is a rational smooth projective variety, then \( H^3(X, \mathbb{Z}) \) is torsion-free.

So we have the Artin–Mumford invariant

\[
H^3(X, \mathbb{Z})_{\text{tors}}
\]

as a candidate to detect non-rationality.

But \( H^3(X, \mathbb{Z})_{\text{tors}} = 0 \) for all interesting examples.
Definition (Voisin, 2015)

$X$ admits a decomposition of the diagonal if in $\text{Chow}(X \times X)$,

$$[\Delta] \sim \{x\} \times X + \alpha$$

for some $\alpha$ supported on $X \times Z$ for $Z \subsetneq X$. 

![Diagram of the diagonal decomposition](image-url)
Decomposition of the diagonal

Theorem (Voisin, 2015)

1. $X$ rational $\implies$ $X$ admits a decomp. of the diagonal.
2. $X$ admits decomp. of the diagonal $\implies$ $H^3(X, \mathbb{Z})_{\text{tors}} = 0$.
3. If $X_t$ is a family of varieties such that some $X_{t_0}$ does not admit a decomp. of the diagonal, then neither does $X_t$ for almost all $t$.

For example, $X_t = \{x^4 + y^4 + z^4 + w^4 - txyzw = 0\}$. 
Decomposition of the diagonal

New technique for non-rationality theorems:

1. Consider a family $X_t$.
2. Find a $t_0$ such that $X_{t_0}$ does not admit a decomposition of the diagonal (for example, show $H^3(X_{t_0}, \mathbb{Z})_{\text{tors}} \neq 0$).
3. Theorem: Almost all $X_t$ are not rational!

- Very general quartic double solids are not rational (Voisin, 2015).
- Rationality is not deformation invariant (Hassett–Pirutka–Tschinkel, 2016).
- Very general hypersurfaces in $\mathbb{P}^{n+1}$ of degree $d \geq \log_2 n + 2$ are not rational (Schreieder, 2018).
There’s a lot more...

1. Kodaira problem,
2. Green’s conjecture for canonical curves,
3. Chow rings of K3 surfaces,
4. Many questions related to the Hodge conjecture.
Thank you!