GIT STABILITY OF FIRST SYZYGIES OF CANONICAL CURVES

CONTENTS

1. MODELS OF M_g FROM GIT: HILBERT POINTS (55)

Consider a smooth curve C embedded in a projective space $\mathbf{P}V$ by a complete pluricanonical linear system:

$$C \subset \mathbf{P}V, \quad V = H^0(C, k\omega).$$

Since this embedding is canonically associated to the curve, all the extrinsic projective data of this embedded curve is in fact intrinsic to the curve. We can use this data to capture the moduli of the curve. This idea gives us recipes to construct several models of M_g.

For example, we can look at homogeneous polynomials of a given degree contained in the ideal of C. In other words, we can look at the subspace

$$H^0(I_C(m)) \subset \operatorname{Sym}_m V.$$

Fixing an identification $V \cong \mathbf{C}^N$, we can interpret this as a point in the appropriate Grassmannian:

$$[H^0(I_C(m)) \subset \operatorname{Sym}_m \mathbf{C}^N] \in \mathbf{Gr}.$$

Call this point the *mth Hilbert point* of C, denoted by $[C]_{k,m}$. Define $\overline{\mathrm{Hilb}}_{k,m} \subset \mathbf{Gr}$ as the closure of the mth Hilbert points of k-canonically embedded smooth curves of genus g. After taking the quotient by SL_N to eliminate the choice of the basis, we expect to get a birational model of M_g:

$$\overline{\mathrm{Hilb}}_{k,m} /\!/ \mathrm{SL}_N .$$

What can we say about these spaces? If m is sufficiently large, then we can precisely identify these spaces:

$$\overline{\mathrm{Hilb}}_{k,m} /\!/ \mathrm{SL}_N = \begin{cases} \overline{M}_g & k \geq 5,\ m \gg 0, \\ \overline{M}_g^{ps} & k = 3,4,\ m \gg 0, \\ \overline{M}_g^{hs} & k = 2,\ m \gg 0. \end{cases}$$

Beyond this, we do not know much. In fact, it is not even clear a priori that these spaces are non-empty. That they are indeed non-empty (and hence birational to M_g) is a recent result of Alper, Fedorchuk, and Smyth:

Theorem 1.1. *If C is a general curve, then $[C]_{k,m}$ is GIT-stable.*

Although we are too far from being able to describe these GIT quotients, modulo certain operative assumptions, we can make some predictions. Assume, then, that possibly outside from a locus of codimension at least two, every point of $\mathrm{Hilb}_{k,m} /\!/ \mathrm{SL}_N$ represents an at worst nodal curve. Then, a relatively straightforward Grothendieck–Riemann–Roch computation shows that the natural line bundle on this quotient, furnished by its GIT construction, is a multiple of

$$L_{k,m} = \begin{cases} \left(12 - \frac{1}{km^2}\left(4km + 4m + 2\right)\right)\lambda - \delta & \text{if } k > 1 \\ \left(8 + \frac{4}{g} - \frac{2}{m} + \frac{2}{g(m-1)}\right)\lambda - \delta & \text{if } k = 1. \end{cases}$$

Therefore, we can expect that $\overline{\mathrm{Hilb}}_{k,m} /\!/ \mathrm{SL}_N$ is a log-canonical model of $\overline{M}_g$.

To get an idea of where we are, let us place these models on the λ-δ plane. Recall that the canonical divisor of $\overline{M}_g$ is $13\lambda - 2\delta$, which corresponds to the ray with slope 6.5 in this picture. Also recall that the rays with slope greater than 11 represent ample divisors (and hence correspond to $\overline{M}_g$ itself). The models for $k > 1$ give models with slope ranging from 12 to a little under 9, followed by the models for $k = 1$, which give slopes ranging from $8 + 4/g$ to $7 + 6/g$.

This is great, except that $7 + 6/g$ is not quite 6.5! If we we wish to continue this program to get the canonical model, we need to dig deeper.

2. Syzygy points (45)

The proposed idea for how to go further is originally due to Sean Keel. He suggested going beyond the generators of the defining ideal to the relations among them, and then to relations among the relations, and so on, namely to the syzygies. Following his idea, let us define the Syzygy points analogous to the Hilbert points. This definitions is via Koszul cohomology. Let us quickly recall its definition.

Let F be sheaf on $\mathbf{P}V$. Consider the complex

$$\cdots \to \wedge^{p+1} V \otimes H^0(F(q-1)) \to \wedge^p V \otimes H^0(F(q)) \to \wedge^{p-1} V \otimes H^0(F(q+1)) \to \dots.$$

Define $K_{p,q}$ to be the cohomology of this complex. It is easy to check that

$$K_{p,q} = \mathrm{Tor}_p(F)_{p+q},$$

namely, it is the degree $(p+q)$ piece of the pth group of syzygies.

What are these groups for a canonically embedded curve, that is, in the case $V = H^0(C, \omega)$ and $F = O_C$? We know that the minimal free resolution of O_C has the form

$$O \leftarrow O(-2)^* \leftarrow O(-3)^* \leftarrow \cdots \leftarrow O(-c)^* \leftarrow O(-1-c)^* \leftarrow \cdots \leftarrow O(-g+1+c)^*$$
$$O(-2-c) \leftarrow \cdots \leftarrow O(-g+c)^* \leftarrow O(-g-1+c) \leftarrow \cdots \leftarrow O(-g+1) \leftarrow O(-g-1) \leftarrow 0.$$

We have the following amazing conjecture about this number c:

Conjecture 2.1 (Green–Lazarsfeld)**.** *In the above setup,* $c = \mathrm{Cliff}(C)$.

For a general curve, this is a theorem:

Theorem 2.2 (Voisin). *If C is general, then $c = \lfloor (g-1)/2 \rfloor$.*

In fact, we know more: thanks to the work of Voisin, Farkas and Aprodu, if C lies on a K3 surface, then the number c is indeed the Clifford index.

Moreover, the whole resolution is symmetric, so we may restrict to $p < (g-1)/2$. Then all the $K_{p,q}$ except $K_{p,1}$ and $K_{p,2}$ vanish, and (conjecturally) for p greater than the Clifford index, $K_{p,2}$ vanishes as well.

Let us focus, then, at the beginning of the Koszul complex (with a slight shift in notation):

$$K : \wedge^{p+2} H^0(\omega) \to \wedge^{p+1} H^0(\omega) \otimes H^0(\omega) \to \wedge^p H^0(\omega) \otimes H^0(2\omega) \to \dots.$$

Set

$$\begin{aligned} V &= H^0(\omega), \\ \Gamma_p(V) &= \wedge^{p+1} V \otimes V / \wedge^{p+2} V, \\ Q_p &= \ker(K_2 \to K_3), \end{aligned}$$

Truncating the Koszul complex, we get the exact sequence

$$0 \to K_{p+1,1} \to \Gamma_p V \to Q_p \to K_{p,2} \to 0.$$

Definition 2.3. Suppose $K_{p,2} = 0$. Then, the *pth syzygy point* $[\mathrm{Syz_p C}]$ is the point in $\mathbf{Gr}(*, \Gamma_p V)$ given by the sequence

$$0 \to K_{p+1,1} \to \Gamma_p \to Q_p \to 0.$$

3. THE MAIN THEOREM (35)

Let $\overline{\mathrm{Syz}}_p \subset \mathbf{Gr} = \mathbf{Gr}(r_p, \Gamma_p \mathbf{C}^g)$ be the closure of the syzygy points. We now have new candidates for the models of M_g, namely

$$\overline{\mathrm{Syz}}_p / / \mathrm{SL}_g .$$

Proposition 3.1. *Suppose $\overline{\mathrm{Syz}}_p / / \mathrm{SL}_g$ is non-empty and away from a locus of codimension at least two, its points represent at worst nodal curves. Then the GIT-induced ample line bundle on this quotient is a multiple of*

$$L_p = \left(8 + \frac{4}{g} - \frac{(g-2)(g-1)}{g(g-p-1)} \right) \lambda - \delta.$$

The proof is a standard GRR computation using the Koszul complex, just as in the case of Hilbert points.

Suppose $p = 0$. Then $L_0 = 7 + 6/g$. This is expected. After all, in this case

$$\Gamma_p V = V \otimes V / \wedge^2 V = \mathrm{Sym}_2 V,$$

and the subspace $K_{p-1,1} = H^0(I(2))$. So this is just the last Hilbert point. However, note that, for higher p, the slopes of these divisors do cover the range from $7 + 6/g$ to 6.5, in fact, asymptotically, they go up to slope 6. Therefore, we are compelled to study these GIT quotients. Our main result is a modest first step in this direction.

Theorem 3.2 (Main). *Let g be odd. Then the GIT quotient of the first syzygies, namely* $\mathrm{Syz}_1 / / \mathrm{SL}_g$ *is non-empty. In other words, if C is general, then* $[\mathrm{Syz}_1(C)]$ *is GIT-stable.*

4. Sketch of the proof (23)

4.1. **Kempf's Criterion.** The idea of the proof is the same as in Alper, Fedorchuk, and Smyth's paper. The details get much more complicated, however, keeping us from extending the result for the higher syzygies.

It suffices to exhibit a GIT-semistable point in $\overline{\mathrm{Syz}}$. In particular, it suffices to exhibit a smoothable, canonically embedded curve C whose syzygy point is semistable.

Which C should we choose? Here the basic idea is due to Swinarski and Morrison, based on a result of Kempf.

Proposition 4.1. *Suppose a reductive group G acts on C such that $V = H^0(C, \omega)$ is a multipicity free representation, say*

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_r.$$

Then the Hilbert/Syzygy points of C are (semi)stable if and only if they are (semi)stable with respect to all one parameter subgroups $\mathbf{G}_m \to \mathrm{SL}\, V$ that preserve the direct sum decomposition.

The idea is to choose C whose automorphism group is rich enough to restrict the choice of one-parameter subgroups as much as possible. Of course, if we restrict to smooth curves C, then G can only be a finite group, but if we allow C to be highly singular, even non-reduced, then there are many more possibilities. Following, Alper, Fedorchuk, and Smyth, we work with ribbons.

4.2. **Ribbons (18).** A ribbon, or more precisely, a rational ribbon is a double structure on $\mathbf{P}^1$, it is a scheme R whose reduction is $\mathbf{P}^1$ and which locally looks like $\mathbf{A}^1 \times \mathrm{Spec}\, \mathbf{C}[\epsilon]/\epsilon^2$. Let R be a ribbon of arithmetic genus g. We have the sequence

$$0 \to O_{\mathbf{P}^1}(-g-1) \to O_R \to O_{\mathbf{P}^1} \to 0.$$

Thus, O_R is an extension of $O_{\mathbf{P}^1}$ by $O_{\mathbf{P}^1}(-g-1)$. These are classified by

$$\mathrm{Ext}^1(\Omega_{\mathbf{P}^1}, O_{\mathbf{P}^1}(-g-1)) \cong H^0(O_{\mathbf{P}^1}(g-3))^\vee.$$

More explicitly, R is obtained by gluing $\mathbf{C}[x,\epsilon]/\epsilon^2$ and $\mathbf{C}[y,\eta]/\eta^2$ by

$$x^{-1} = y + p(y)\eta$$
$$\epsilon = y^{-g-1}\eta,$$

where $p(y) \in \langle y^{-1}, \ldots, y^{-(g-2)} \rangle$.

Notice that if $p(y)$ is a monomial, then R admits a $\mathbf{G}_m$ action. In particular, let $g = 2k+1$. Then there is a distinguished choice of a monomial—the most balanced one

$$p(y) = y^{-k}.$$

The $\mathbf{G}_m$ action is

$$t\colon x \mapsto tx, \quad \epsilon \mapsto t^{k+1}\epsilon$$
$$: y \mapsto t^{-1}y, \quad \eta \mapsto t^{-k-1}\eta.$$

Ribbons have a canonical line bundle (since they are Gorenstein), a canonical embedding (if they are not "hyperelliptic", that is if $p(y) \neq 0$), and are limits of canonical curves. For the balanced ribbon R, the sections of the canonical written explicitly are:

$$H^0(\omega) \cong \langle 1, x, x^2, \dots, x^k \\ x^{k+1} + \epsilon, x^{k+2} + x\epsilon, \dots, x^{2k} + kx^{k-1}\epsilon\rangle.$$

Note that these sections are eigenvectors of the $\mathbf{G}_m$ action with distinct weights. That is, $H^0(\omega)$ is a direct sum of g distinct characters of $\mathbf{G}_m$.

It is easy to check that the balanced ribbon R satisfies $K_{1,2} = 0$ and hence has a well-defined syzygy point.

Proposition 4.2. $[\mathrm{Syz}_1 \mathrm{R}]$ *is semi-stable.*

The beauty is that we only need to check that $[\mathrm{Syz}_1 \mathrm{R}]$ is semi-stable with respect to the *diagonal* tori, that is, on $\mathbf{G}_m$'s acting by scaling the canonical sections $x_0, \dots x_{2k}$. This is ultimately a purely combinatorial question, which is not to say that it is *easy*. Most of the work in our paper goes into checking this linear-algebraic/combinatorial condition. It would be great to have systematic tools to attack these kinds of questions. In his talk, David Swinarski will explore some systematic approaches.

4.3. **Monomial Bases (10).** In any case, let me describe what is involved in the remaining details. Let $T \subset \mathrm{SL}\, V$ be the diagonal torus, where $V = H^0(\omega) = \langle x_0, \dots, x_{2k}\rangle$. The (co)syzygy point of R is the point in the grassmannian of $r = (g-1)(3g-5)$ dimensional quotients of $\Gamma_1 V$ given by

$$\Gamma_1 V = \wedge^2 V \otimes V / \wedge^3 V \to Q_1 \to 0,$$

where

$$Q_1 = \ker(H^0(R, \omega) \otimes H^0(R, 2\omega) \to H^0(R, 3\omega)).$$

Going to the Plucker coordinates, we get the point in the projective space

$$\zeta \equiv [\wedge^r \Gamma_1 V \to \wedge^r Q_1 \to 0].$$

Now, note that $\wedge^r \Gamma_1 V$ is a representation of T, say

$$\wedge^r \Gamma_1 V = \oplus \mathbf{C}_\chi,$$

where the χ's are characters of T. This decomposition gives homogeneous coordinates on $\mathbf{P}(\wedge^r \Gamma_1 V)$ with a character associated to each coordinate. The point ζ is T-semistable if and only if for every $\mathbf{G}_m \to T$, the weigts of the non-zero coordinates of ζ are not all positive or negative. This is equivalent to saying that the characters corresponding to non-zero coordinates of ζ contain the trivial character in their convex hull.

So, for the proof, we must exhibit enough non-zero coordinates of ζ to surround the origin. Note that if $s_1, \dots, s_r \in \Gamma_1 V$ are "monomials" $(x_a \wedge x_b \otimes x_c)$ that project onto a basis of Q_1, then $s_1 \wedge \cdots \wedge s_r$ is a non-zero coordinate, whose associated character is the sum of the characters associated to the s_i's. What we must do, then, is to find monomial bases $B_1, B_2, \ldots$ of Q_1 such that their characters $\chi(B_1), \chi(B_2), \ldots$ contain $(0, \ldots, 0)$ in their convex hull.

Since we know $H^0(R, i\omega)$ very well, we are able to do this explicitly. In fact, we can write down three such bases, which suffice. There are some heuristics that go into choosing these bases, thanks to Maksym's earlier experience. Somewhat surprisingly, the purely linear algebraic fact that they are indeed bases was quite difficult to pin down.

In any case, that's done now, so there is now a birational model of slope $7 + 5/g$, waiting to be explored.