Outline of talks :— $\boxed{\text{Talk 2}}$

- What is a Bridgeland stability condition.
  - Definition
  - How choosing a ♡ allows to construct one
  - Examples
  - The stability manifold.
  - $\mathbb{C}$ action.
  - Mass of an object

- $\mathcal{T}$ = Zig Zag category.

- The projective embedding (for a compactification)
  - motivation from Teichmüller theory
  - Definition of the map.
  - injectivity.

- Boundary
  - map from objects.
  - The image lies in the closure.

- Main conjectures — Generality?
  - precompactness
  - homeomorphic embedding.
  - closure = stab/$\mathbb{C}$ ∪ closure of obj.
  - Closure is a manifold with boundary.

- q-analog.

Compactifying Stab : A_2 case

§1. The Category

$$\mathcal{C} = K(\mathrm{Proj}\, Z(A_2))$$

$$A_2 = \circ - \circ \quad \rightsquigarrow \quad \begin{tikzcd} 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "b"] \end{tikzcd}$$

$Z(A_2) =$ Path Algebra / (aba, bab)

$e_1 =$ empty path at 1 $\qquad e_1^2 = e_1$

$e_2 =$ empty path at 2 $\qquad e_2^2 = e_2$

$$\left.\begin{array}{l} P_1 = Ze_1 \\ P_2 = Ze_2 \end{array}\right\} \text{Indecomposable projective.}$$

$$\mathrm{Hom}^n_{\mathcal{C}}(P_i, P_i) = \mathrm{Hom}_{\mathcal{C}}(P_i, P_i[n]) = \begin{cases} \mathbb{C} & n = 0 \text{ or } 2 \\ 0 & \text{otherwise.} \end{cases}$$

$$\underset{i \neq j}{\mathrm{Hom}^n_{\mathcal{C}}}(P_i, P_j) = \begin{cases} \mathbb{C} & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

$\mathcal{C}$ is 2-CY

is Characterised by

- 2-CY
- (classically) generated by P_1 & P_2 satisfying Hom conditions above.

§ The Standard Heart

$\heartsuit$ = Ext. closure of P_1 and P_2.
= Category of "Linear complexes".

Simple objects = P_1, P_2 (spherical)
Two other spherical objects -

$$P_1 \to P_2 = \overset{0}{P_1} \to \overset{1}{P_2\{1\}}$$
$$= \mathrm{Cone}\,(P_1 \to P_2[1])$$
$$= \sigma_{P_2}^{-1}(P_1)$$

$$P_2 \to P_1 = \mathrm{Cone}\,(P_2 \to P_1[1]).$$
$$= \sigma_{P_1}^{-1}(P_2).$$

§ Spherical Twists

G = Spherical twist group $\cong$
$$\langle \sigma_1, \sigma_2 \rangle / \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$$

$\cong$ 3 Strand Braid group B_3.

$$(\sigma_1\sigma_2)^3 = [1].$$
$$G/(\sigma_1\sigma_2)^3 = PSL_2(\mathbb{Z}) = \overline{G}$$
$$\sigma_1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \sigma_2 \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

§ Spherical Objects

$$\text{Sphericals} = G \cdot P_1 = G \cdot P_2$$

$$S = \text{Sphericals} / \text{Shift}$$

$$\overline{G} \text{ set } S = PSL_2(\mathbb{Z}) \text{ set } \mathbb{P}^1(\mathbb{Q})$$

$$P_1 \longleftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

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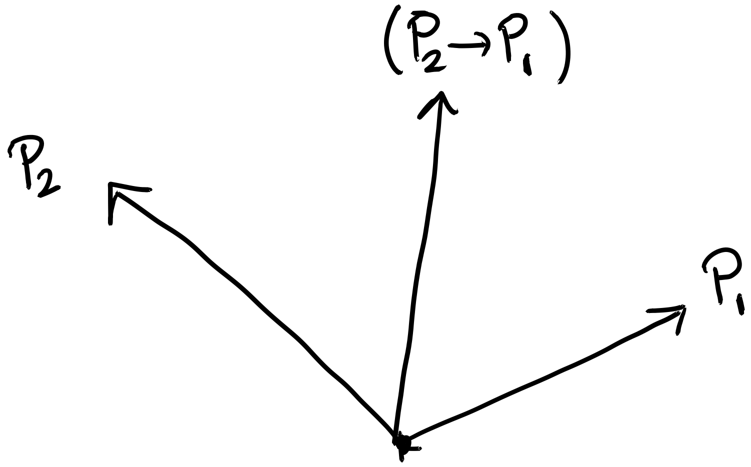
§ Stability conditions

$$\text{Stab} = \text{Heart} + Z.$$
$$K_0(\mathcal{C}) = \mathbb{Z}[P_1] \oplus \mathbb{Z}[P_2].$$

$$\text{Heart} = \heartsuit.$$

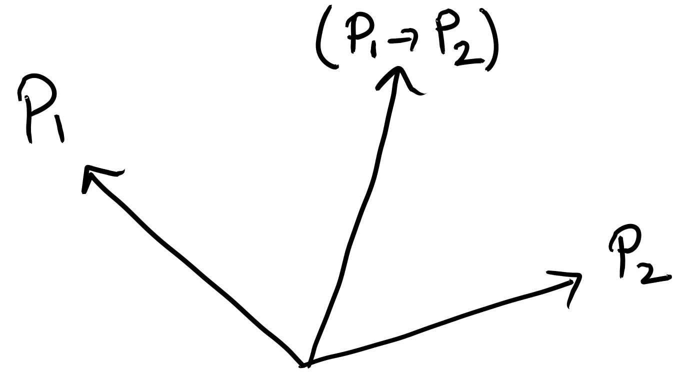
Z:



1

$$P_2 \to (P_1 \to P_2) \to P_1$$
$$(P_1 \to P_2) \sim_{HN} P_1 + P_2.$$

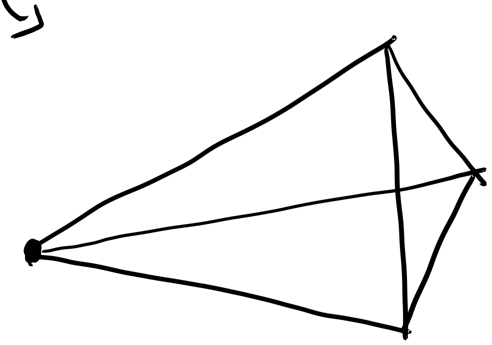
Z:



$$P_2 \to P_1 \sim_{HN} P_1 + P_2 .$$

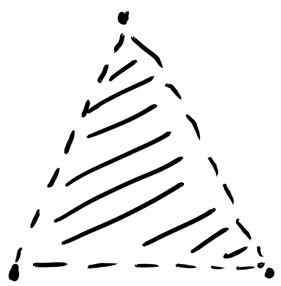
Stab. cond. of type 1 is determined uniquely, up to rotation, by

$$\left.\begin{array}{l} m(P_1) \\ m(P_2) \\ m(P_1 \to P_2) \end{array}\right\} \text{ positive real numbers satisfying the triangle inequality.}$$



$\subset \mathbb{R}^3.$

type 1 up to rotation & scaling $\cong$

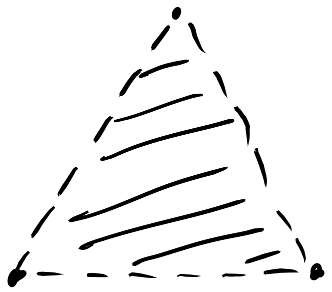


$\subset \mathbb{P}^2(\mathbb{R})$

$$\tau \longmapsto [m_\tau(P_1) : m_\tau(P_2) : m_\tau(P_2 \to P_1)]$$

Similarly

type (2) up to rotation & scaling $\cong$



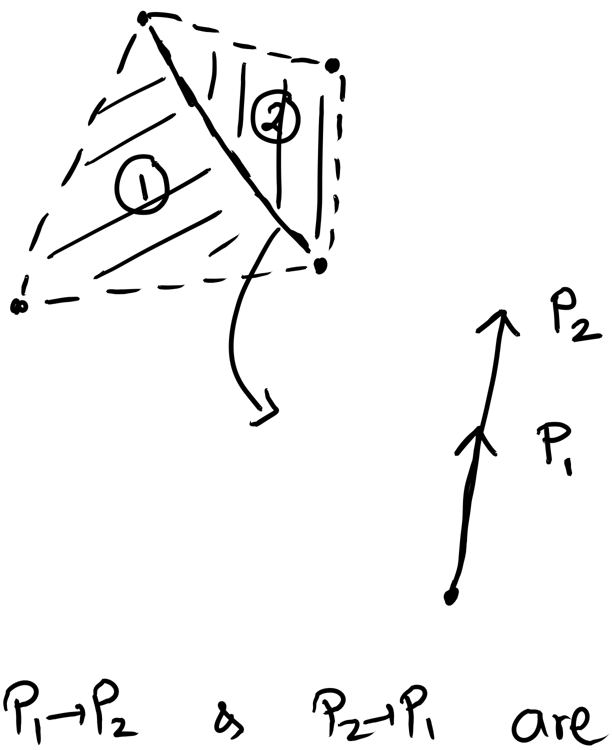
$\hookleftarrow \mathbb{P}^2(\mathbb{R})$

$\tau \longmapsto [m_\tau(P_1) : m_\tau(P_2) : m_\tau(P_1 \to P_2)]$

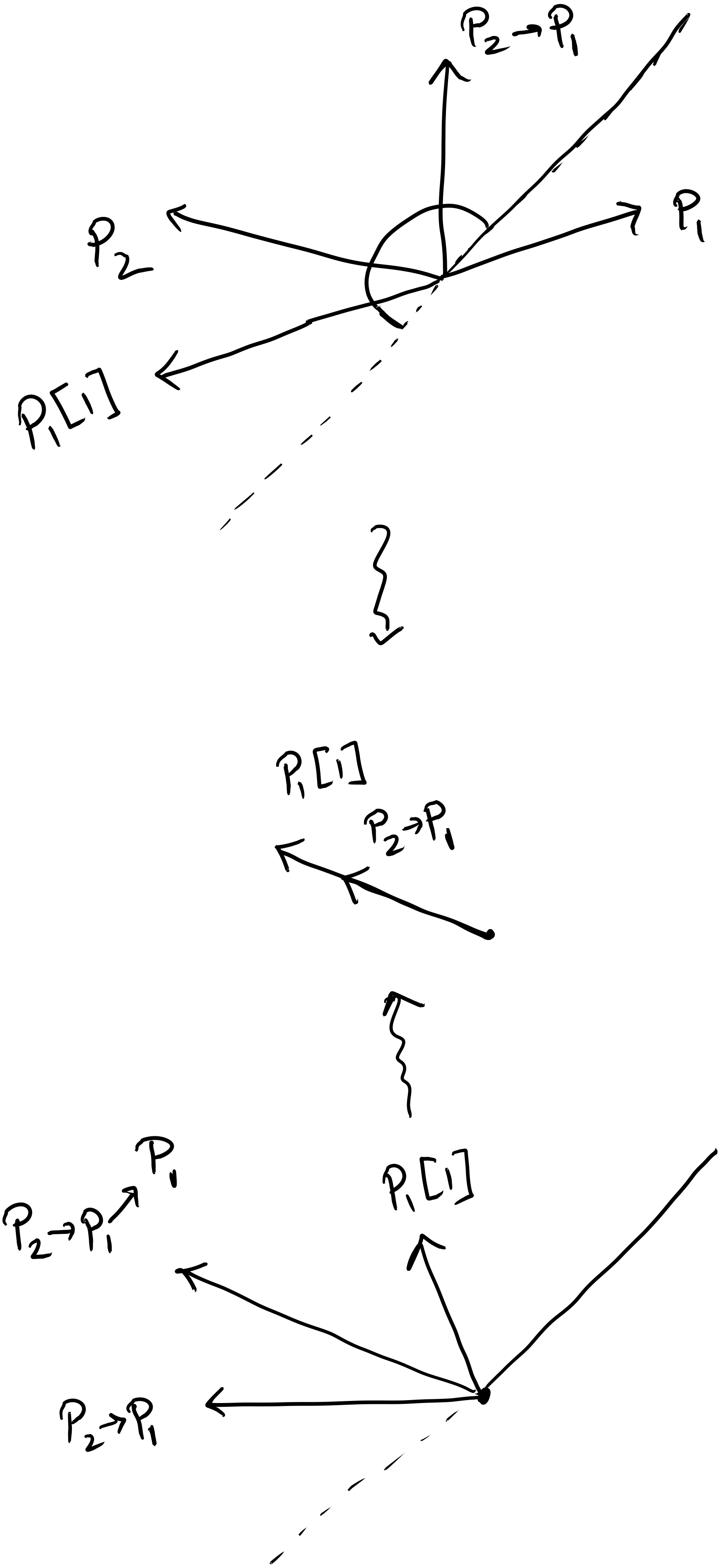
Type (1) or type (2)

$\tau \mapsto \mathbb{P}^3(\mathbb{R})$

$[m_\tau(P_1) : m_\tau(P_2) : m_\tau(P_2 \to P_1) : m_\tau(P_1 \to P_2)]$



Both $P_1 \to P_2$ & $P_2 \to P_1$ are semistable



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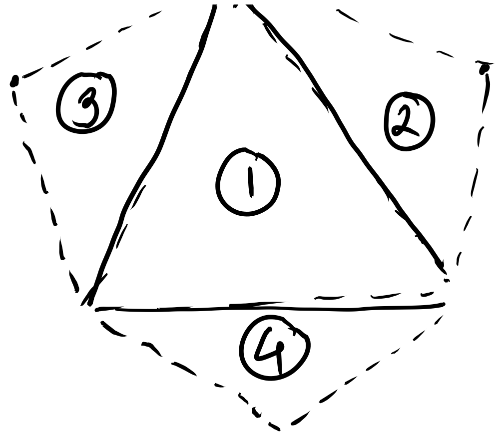


$\subseteq \mathbb{P}^5(\mathbb{R})$

<u>Obs</u>: Type ② $= \sigma_1$ (Type ①)

③ $= \sigma_1^{-1}$(①)

④ $= \sigma_2$ (①)

<u>Thm</u> ① $\mathrm{Stab}(A_2)/\mathbb{C} \hookrightarrow \mathbb{P}(\mathbb{R}^5)$

is a homeomorphism onto its image

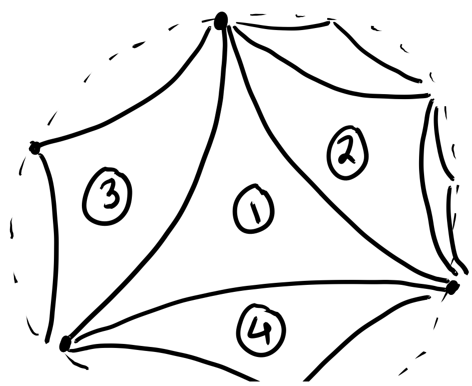
$\boxed{\text{The Stability manifold}}$ ($\mathrm{Stab}(A_2)$ = conn. comp containing std)

<u>Thm</u> (Bridgeland, Sutherland, Qiu; —)

We have a homeomorphism

$\mathrm{Stab}(A_2)/\mathbb{C} \cong$ Open unit disk

compatible with $PSL_2(\mathbb{Z})$ actions such that type ① $\cong$ An ideal triangle.



Thm (—)

② The homeomorphism

$$\text{Open disk} \xrightarrow{\sim} \mathrm{Stab}/\mathbb{C}$$

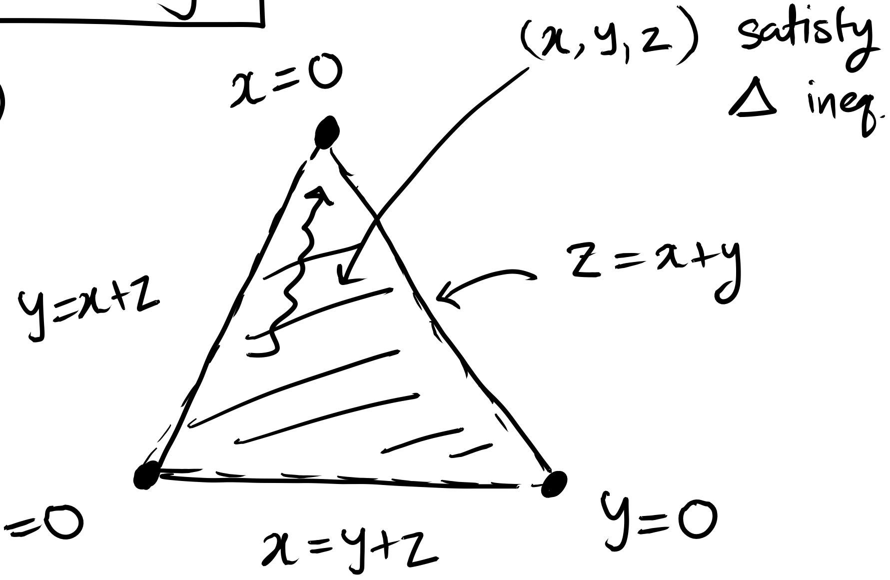
extends to a homeomorphism

$$\text{Closed disk} \xrightarrow{\sim} \overline{\mathrm{Stab}/\mathbb{C}}$$

and the boundary S^1 = closure of hom functionals.

The Boundary

Type ①



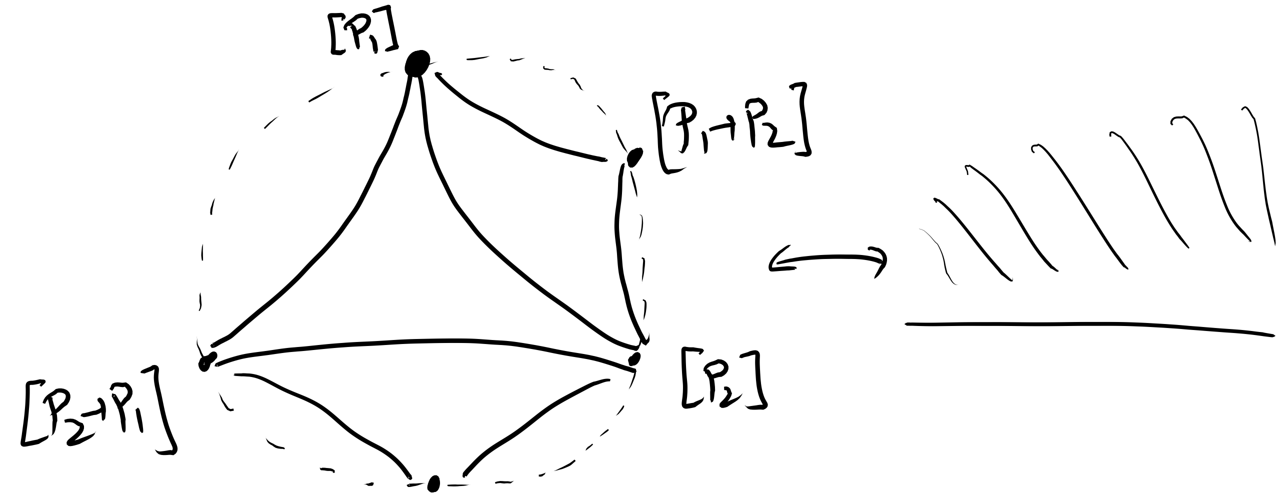
$m_\tau(P_1) = x, \quad m_\tau(P_2) = y \quad , m_\tau(P_2 \to P_1) = z$

Say $(x, y, z) \rightsquigarrow (0, 1, 1)$.

$$m_\tau(X) = \#(P_2) + \#(P_2 \to P_1) \text{ in HN}$$
$$= \# P_2 \text{ in minimal complex}$$

$$= \overline{\hom}(X, P_1) \qquad \text{(Prop.)}$$

Thus as $(x, y, z) \rightsquigarrow (0, 1, 1)$
$\tau \rightsquigarrow \overline{\hom}(-, P_1)$ in $\mathbb{P}^\infty$.



The rational points of the boundary
||
hom functionals. $\leftrightarrow$ objects in $\mathcal{C}$

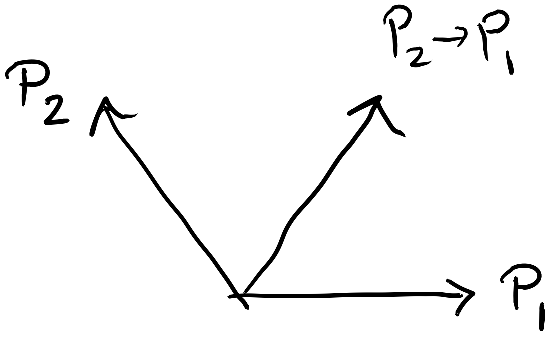
Non-rational points = ?
(certain functionals)
$\updownarrow$
Categorical interpretation ?

σ_T = Rotation.
Illustrates Asilata's Statement.

Nielson-Thurston Classification

① Periodic — Has an interior fixed point.
e.g. $\sigma_1 \sigma_2$

Fixes



② Reducible - Has no interior fixed points but a unique fixed point on the boundary

e.g. σ_1

Fixes $[P_1]$.

③ Pseudo-Anosov - Has no interior fixed pts and two fixed points on the boundary

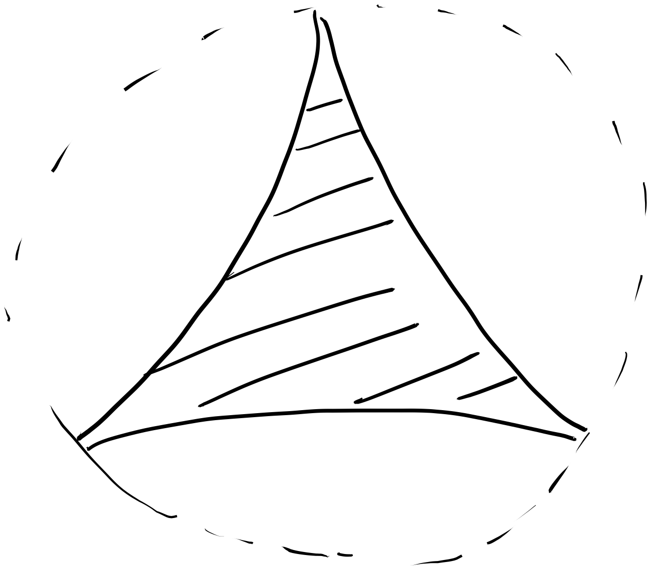
eg. $\sigma_1 \sigma_2^{-1}$ $\begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

Fixes $\left[\frac{\sqrt{5}-1}{2} : 1 \right]$ & $\left[\frac{-\sqrt{5}-1}{2} : 1 \right]$.

attracting repelling.

"Pair of transverse foliations"

$\boxed{q\text{-analog.}}$



$i: PSL_2(\mathbb{Z}) \subset PSL_2(\mathbb{R}) \curvearrowright \text{Disk}$

$i_q: PSL_2(\mathbb{Z}) \hookrightarrow PSL_2(\mathbb{R})$

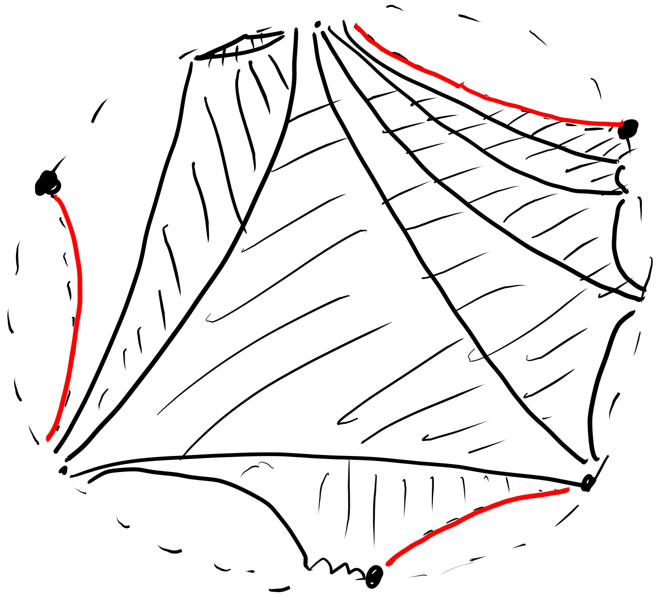
$i: \sigma_1 \mapsto \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \qquad \sigma_2 \mapsto \begin{pmatrix} & 1 \\ -1 & 1 \end{pmatrix}$

$i_q: \sigma_1 \mapsto \begin{pmatrix} & \\ & \end{pmatrix} \qquad \sigma_2 \mapsto \begin{pmatrix} & \\ & \end{pmatrix}$

$PSL_2(\mathbb{Z}) \underset{i_q}{\curvearrowright} \text{Disk}.$

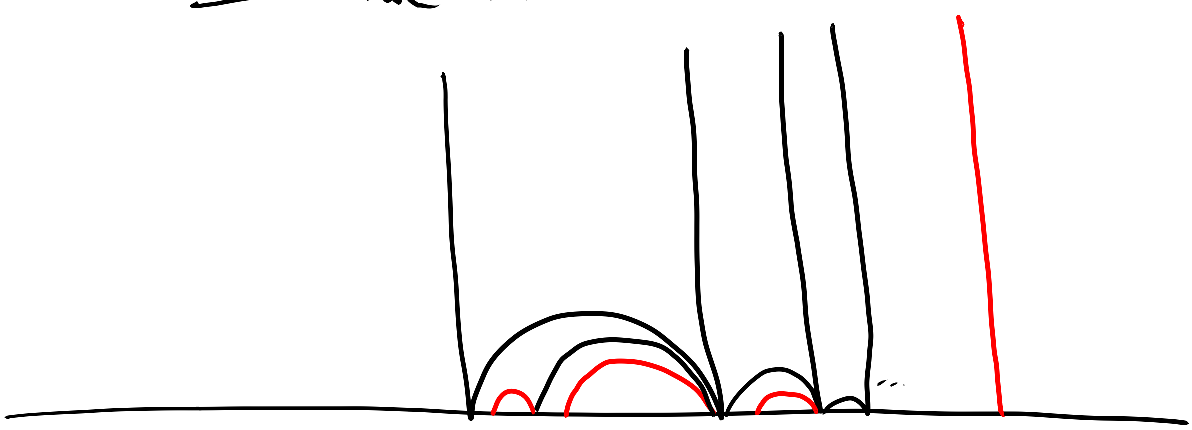
$$D = \bigcup_{\tau \in PSL_2(\mathbb{Z})} i_q(\tau) \cdot \Delta \cong \text{Disk}$$

$$\cong \text{Stab}/\mathbb{C}$$



$$\begin{tikzcd} D \arrow[r, hook] \arrow[d, "2"] & \overline{D} \arrow[d, "2"] \arrow[r, phantom, "\subsetneq"] & \overline{\mathrm{Disk}} \\ \mathrm{Stab}/\mathbb{C} & \overline{\mathrm{Stab}}/\mathbb{C} & \\ \mathbb{RP}^\infty \arrow[u, hook] \arrow[r, equal] & \mathbb{RP}^\infty \arrow[u, hook] & \end{tikzcd}$$

S not dense in $\overline{D}$.



Compactifying Stab : A_2

The A_2-Category

$\mathcal{T} = \mathcal{T}(\circ\!-\!\!\!-\!\circ)$

- Triangulated $\mathbb{C}$-linear
- 2CY i.e. $\mathrm{Hom}(A,B) = \mathrm{Hom}(B, A[2])^*$
- Classically generated by P_1 & P_2

$$\mathrm{Hom}^n(P_i, P_i) = \begin{cases} \mathbb{C} & n = 0, 2 \\ 0 & \text{else} \end{cases}$$

$$\mathrm{Hom}^n(P_i, P_j) = \begin{cases} \mathbb{C} & n = 1 \\ 0 & \text{else.} \end{cases}$$

Spherical twist group

$$G = \langle \sigma_1, \sigma_2 \rangle \;/\; \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$$

$\cong$ 3 strand braid group.

$\sigma_i \longmapsto \sigma_{P_i}$

$$(\sigma_1\sigma_2)^3 = [-2] \in Z(G)$$

$$G/(\sigma_1\sigma_3)^3 \cong PSL_2(\mathbb{Z})$$

$\sigma_1 \mapsto \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ $\qquad$ $\sigma_2 \mapsto \begin{pmatrix} 1 & \\ -1 & 1 \end{pmatrix}$

Spherical objects

$$\text{Sphericals} = G \cdot P_1$$

$$S = \text{Spherical / shift}$$
$$= PSL_2(\mathbb{Z}) \cdot [P_1]$$

$$\text{Stabilizer of } [P_1] = \langle \sigma_1 \rangle$$

so $S \cong \mathbb{P}^1(\mathbb{Q})$ as a $PSL_2(\mathbb{Z})$ set.

Standard Heart

$\heartsuit$ = Ext. closure of P_1 & P_2
= Finite length Abelian category with two simples, P_1 & P_2

Two other spherical obj.

" $P_1 \to P_2$ " = Unique ext^n of P_1 by P_2
= $\sigma_1(P_2)$

" $P_2 \to P_1$ " = Unique ext^n of P_2 by P_1
= $\sigma_2(P_1)$

Stability conditions.

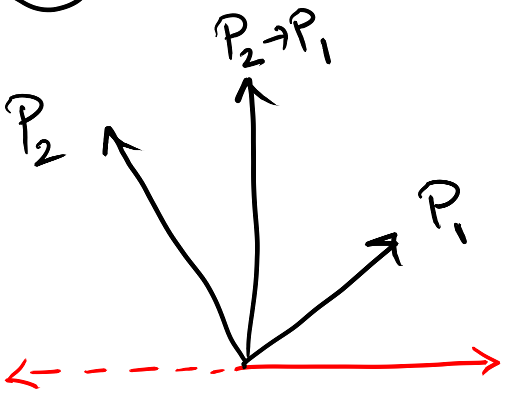
$$\mathrm{Stab} = \mathrm{Heart} + \mathrm{Charge}$$

$$\mathrm{Heart} = \heartsuit$$

$$\mathrm{Charge} = Z : K_0(\mathcal{T}) \longrightarrow \mathbb{C}$$

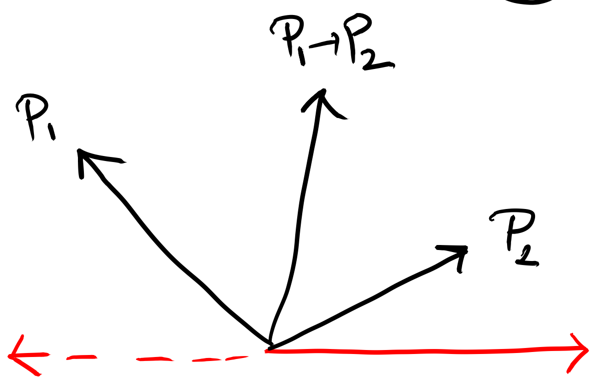
$$K_0(\mathcal{T}) = \mathbb{Z}[P_1] \oplus \mathbb{Z}[P_2]$$

①



$(P_1 \to P_2) \sim_{HN} P_1 + P_2$

②



$(P_1 \to P_2) \sim_{HN} P_1 + P_2$

Type ① conditions

τ of type ① determined (up to rotation) by

$$\left.\begin{aligned} x &= m(P_1) \\ y &= m(P_2) \\ z &= m(P_2 \to P_1) \end{aligned}\right\} \quad (x,y,z) \in (\mathbb{R}_+)^3 \text{ satisfy triangle } \leq$$

$\{\text{Type ①} / \text{rot.}\} =$  $\subset \mathbb{R}^3$

$\{\text{Type ①} / \text{rot} + \text{scaling}\} =$  $\subset \mathbb{P}^2(\mathbb{R})$

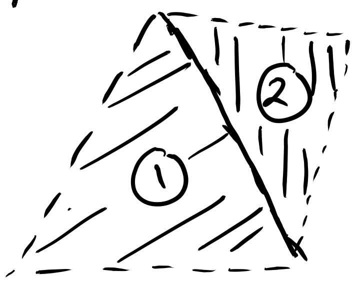
Similarly

$\{\text{Type } ② / \mathbb{C}\} =$  $\subset \mathbb{P}^2(\mathbb{R})$

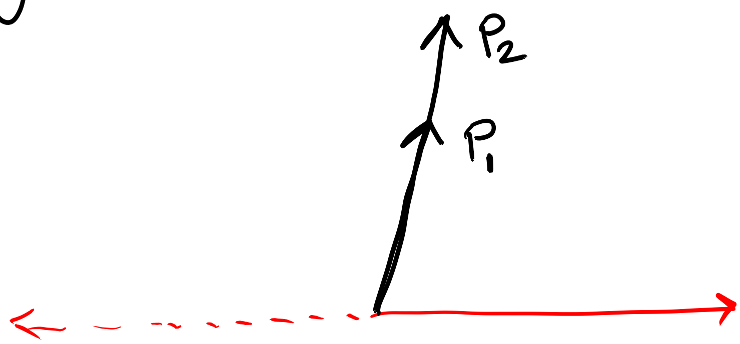
$$\tau \mapsto [m_\tau(P_1) : m_\tau(P_2) : m_\tau(P_1 \to P_2)]$$

two together by $\tau \mapsto [m_\tau(P_1) : m_\tau(P_2) : m_\tau(P_1 \to P_2) : m_\tau(P_2 \to P_1)]$

$$\{ ① \text{ or } ② / \mathbb{C}\} \subset \mathbb{P}^3(\mathbb{R})$$

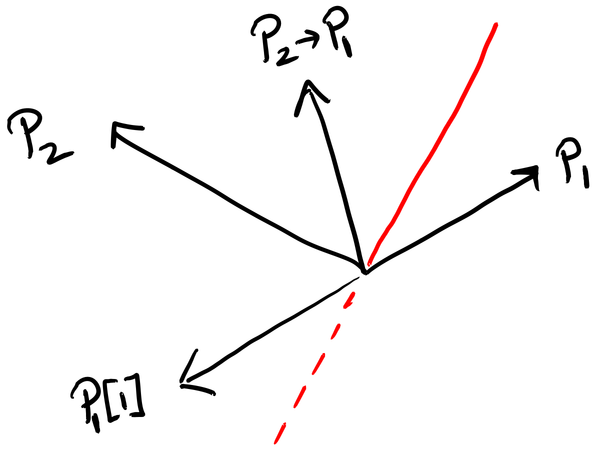


<u>The edge</u>:

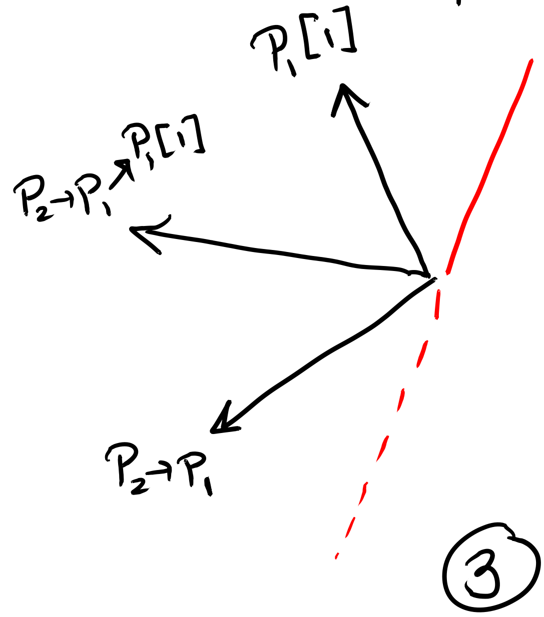


Other two edges of ① ?

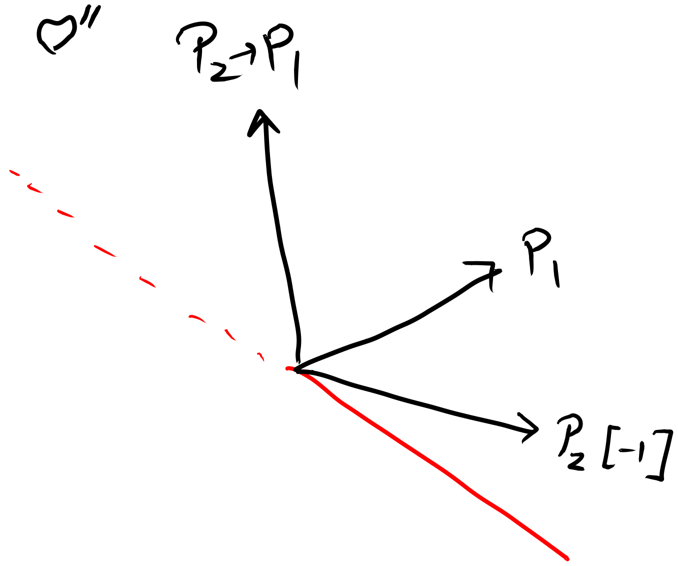
σ'



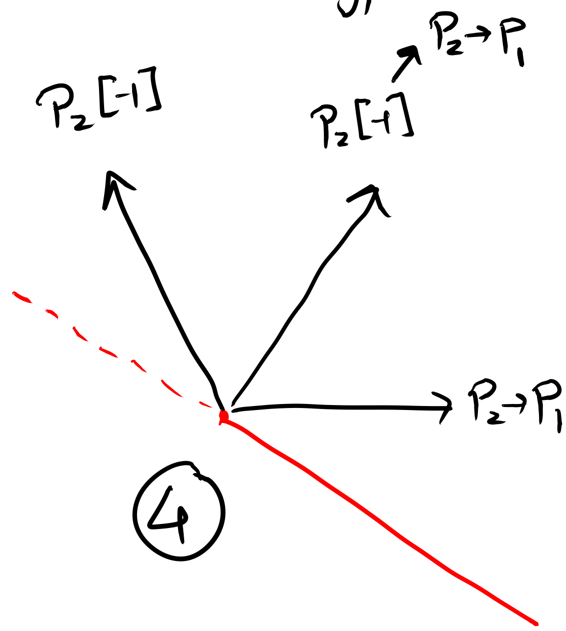
σ' other type



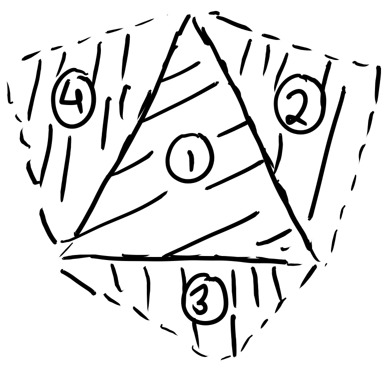
σ''



σ'' other type



In $\mathbb{P}^5(\mathbb{R})$



continue....

Thm(-) We have an embedding

$$\mathrm{Stab}(A_2)/\mathbb{C} \hookrightarrow \mathbb{P}(\mathbb{R}^5)$$

The image is tesselatted by clipped triangles. $PSL_2(\mathbb{Z})$ acts transitively on these triangles.

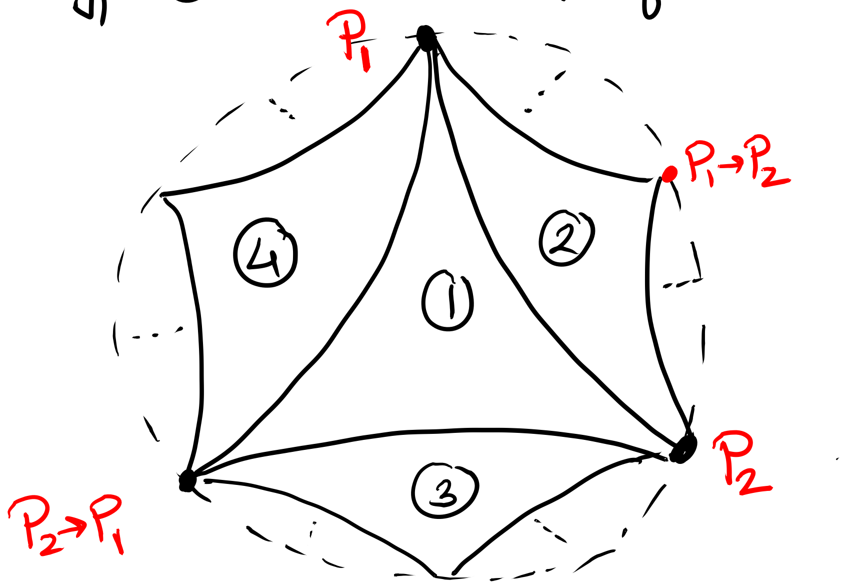
Global Picture ?

Thm (Thomas, Bridgeland-Qiu-Sutherland, Ikeda, —)

There is a $PSL_2(\mathbb{Z})$ equivariant homeomorphism

$$\mathrm{Stab}/\mathbb{C} \cong \text{Open Unit disc in } \mathbb{C}$$

s.t. type ① $\cong$ interior of an ideal triangle



Compactification.

Let $\mathcal{B} \subset \mathbb{P}(\mathbb{R}^S)$ be the homeomorphic image of $\mathrm{Stab}/\mathbb{C}$

$\overline{\mathcal{B}}$ = closure of $\mathcal{B}$.

Thm (—) : The homeomorphism $\mathcal{B} \cong$ unit disk extends to a homeomorphism $\overline{\mathcal{B}} \cong$ closed disk.

The points of $S \subset \overline{\mathcal{B}} \setminus \mathcal{B}$ correspond to the vertices of the ideal triangulation.

Boundary only emerges if you take all Stab; Cannot restrict to finitely many hearts

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In the picture, σ_X = "Rotation about X"

So $\displaystyle\lim_{n\to\infty}(\sigma_X^{\pm n}\tau) = [X]$

as discussed in talk 2.

Nielsen-Thurston classification

$g \in \mathrm{Aut}(\tau) \rightsquigarrow \mathrm{Fix}(g) \subset \overline{\mathrm{Stab}} \rightsquigarrow$ dynamical classification.

① g has an interior fixed pt.
ex. $g = \sigma_1\sigma_2 \rightsquigarrow$ finite order.

② g has a unique fixed pt on boundary
ex. $g = \sigma_1$ or σ_X. $\rightsquigarrow$ "reducible"

③ g has two fixed pts on boundary "pseudo Anosov"
ex. $g = \sigma_1\sigma_2^{-1}$

$$\mathbb{P}^1(\mathbb{R}) \supset \mathbb{P}^1(\mathbb{Q})$$

fixed points are

$$\left[\frac{\sqrt{5}-1}{2} : 1\right] \text{ \& } \left[\frac{-\sqrt{5}-1}{2} : 1\right]$$

Categorical interpretation ??