Syzygies of canonical curves and the geometry of $\overline{M}_g$

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The main question

C a smooth projective curve of genus g.
Take a (pluri)-canonical embedding $C \subset \mathbf{P}^n$.
Consider

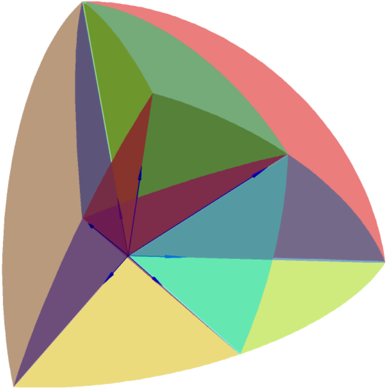
$$\{\text{Space of } C \subset \mathbf{P}^n\} /\!\!/ \mathrm{SL}_{n+1} .$$

Question

How is this quotient related to $\overline{M}_g$?

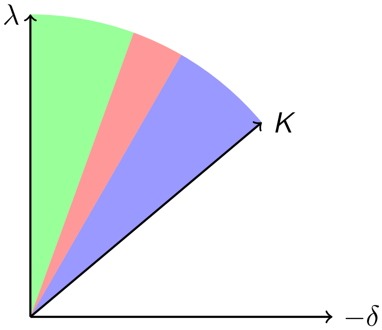
The dream

The Mori chamber decomposition of $\mathrm{Pic}_{\mathbf{Q}}(\overline{M}_g)$ by these spaces.



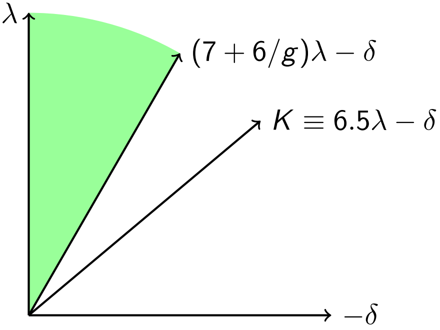
The dream

The Mori chamber decomposition of $\langle \lambda, \delta \rangle \subset \mathrm{Pic}_{\mathbf{Q}}(\overline{M}_g)$ by these spaces.



Space of $C \subset \mathbf{P}^n = \mathbf{P}V$

$$[H^0(\mathcal{I}_C(m)) \subset \mathrm{Sym}^m V] \in \mathbf{Gr}(*, \mathrm{Sym}^m V) /\!\!/ \mathrm{SL}\, V.$$



Space of $C \subset \mathbf{P}^n = \mathbf{P}V$ using syzygies

$$\mathcal{O}_C \leftarrow \mathcal{O} \leftarrow \mathcal{O}(-2)^* \leftarrow \mathcal{O}(-3)^* \leftarrow \ldots.$$

Example (Genus 7):

$$\mathcal{O}_C \leftarrow \mathcal{O} \leftarrow \mathcal{O}(-2)^{10} \leftarrow \mathcal{O}(-3)^{16} \leftarrow \mathcal{O}(-5)^{16} \leftarrow \mathcal{O}(-6)^{10} \leftarrow \mathcal{O}(-8).$$

$K_{p,q}$	0	1	2	3	4	5
0 :	1	.	.	.	.	.
1 :	.	10	16	.	.	.
2 :	.	.	.	16	10	.
3 :	.	.	.	.	.	1

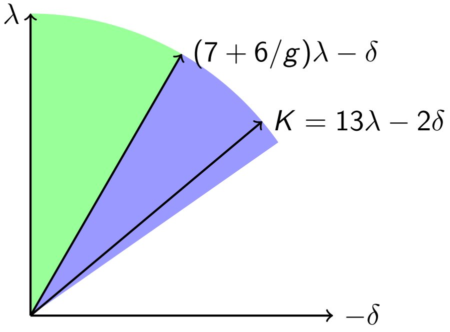
Space of $C \subset \mathbf{P}^n = \mathbf{P}V$ using syzygies

Via the Koszul complex,

$$K_{p,1} \subset \Gamma_p V,$$

where $\Gamma_p V = \wedge^p V \otimes V / \wedge^{p+1} V$.

Take $[K_{p,1} \subset \Gamma_p V] \in \mathbf{Gr}(*, \Gamma_p V) /\!\!/ \mathrm{SL}\, V$.



What is known?

1. Pluricanonical Hilbert quotients for $m \gg 0$ are birational models of $\overline{M}_g$ occupying the first two chambers. [Gieseker, Hassett–Hyeon]
2. Bi-canonical Hilbert quotients are birational to $\overline{M}_g$. Canonical Hilbert quotients are non-empty. [Alper–Fedorchuk–Smyth]
3. For odd g, the first canonical syzygy quotient is non-empty. [D-Fedorchuk-Swinarski]

More in low genera.
[Lee, Jensen, Casalaina-Martin, Laza, Müller,...]

Semistability of the ribbon

Conjecture

For odd g, the pth syzygy point of the balanced canonical ribbon of genus g is semistable.

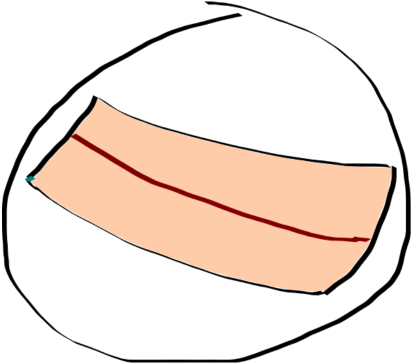
p \ g	7	9	11	13	15
1	✓	✓	✓	✓	✓
2	-	✓	✓	✓	?
3	-	-	✓	?	?
4	-	-	-	?	?
5	-	-	-	-	?

Curves of genus 7

- General

- Tetragonal
  (Codimension 1)

- Unbalanced tetragonal
  (Codimension 2)



Curves of genus 7

- General
  Has the betti table

1	.	.	.	.	.
.	10	16	.	.	.
.	.	.	16	10	.
.	.	.	.	.	1

- Tetragonal
  (Codimension 1)
  1. Has a g^1_4
  2. Has the betti table

1	.	.	.	.	.
.	10	16	3	.	.
.	.	3	16	10	.
.	.	.	.	.	1

- Unbalanced tetragonal
  (Codimension 2)
  1. Has a g^2_6
  2. Has the betti table

1	.	.	.	.	.
.	10	16	9	.	.
.	.	9	16	10	.
.	.	.	.	.	1

  3. Has unbalanced scrollar invariants.

Tetragonal curves and scrollar invariants

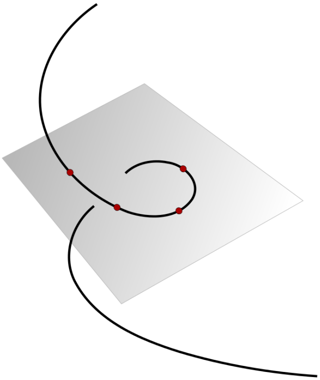
C tetragonal $\implies C \subset \mathbf{P}E$,
where $\pi\colon E \to \mathbf{P}^1$ is rank 3.

In $\mathbf{P}E$, we have $C = X_1 \cap X_2$
where $X_i \in |\mathcal{O}(2) \otimes \pi^*\mathcal{O}(-a_i)|$.

In genus 7, $a_1 + a_2 = 10$.

Generically, $(a_1, a_2) = (5, 5)$
(balanced).

In codim 1, $(a_1, a_2) = (4, 6)$
(unbalanced).

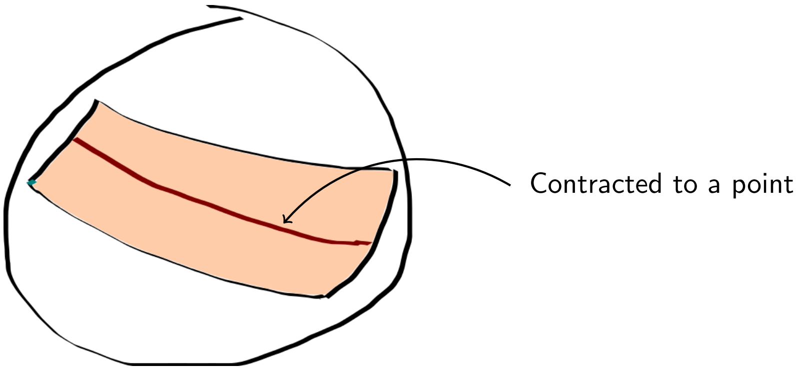


Syzygies in genus 7

Theorem (–)

1. *A general curve of genus 7 has a stable syzygy point.*
2. *A general tetragonal curve has at least a semistable syzygy point.*
3. *A general unbalanced tetragonal curve has a strictly semistable syzygy point. The syzygy points of unbalanced tetragonal curves coincide and are equal to the syzygy point of a del Pezzo surface of degree 6 that contains these curves.*

Syzygy model (speculation)



Proofs

Main idea.

Suppose G acts on C such that $V = H^0(C, \omega_C)$ is a *multiplicity free* G-representation.

Then SL V stability reduces to torus stability.

Checking torus stability is a concrete combinatorial / linear algebraic problem that we can (sometimes) solve in general or verify using a computer for particular cases.

□