APPARENT BOUNDARIES OF PROJECTIVE VARIETIES

Joint with

Anand Patel

Eduard Duryev

A Guessing Game

dim \ deg									
1	1	2	5	14	42	132	429	. . .	Catalan
2		1	1	2	6	22	92	422 . . .	A001181
3			?	?	?				
4			?	?	?				
⋮									

APPARENT BOUNDARY

Smooth projective $X \subset \mathbb{P}^n$ of dim r

General linear $\Lambda \subset \mathbb{P}^n$ of dim $n-r-1$

Project

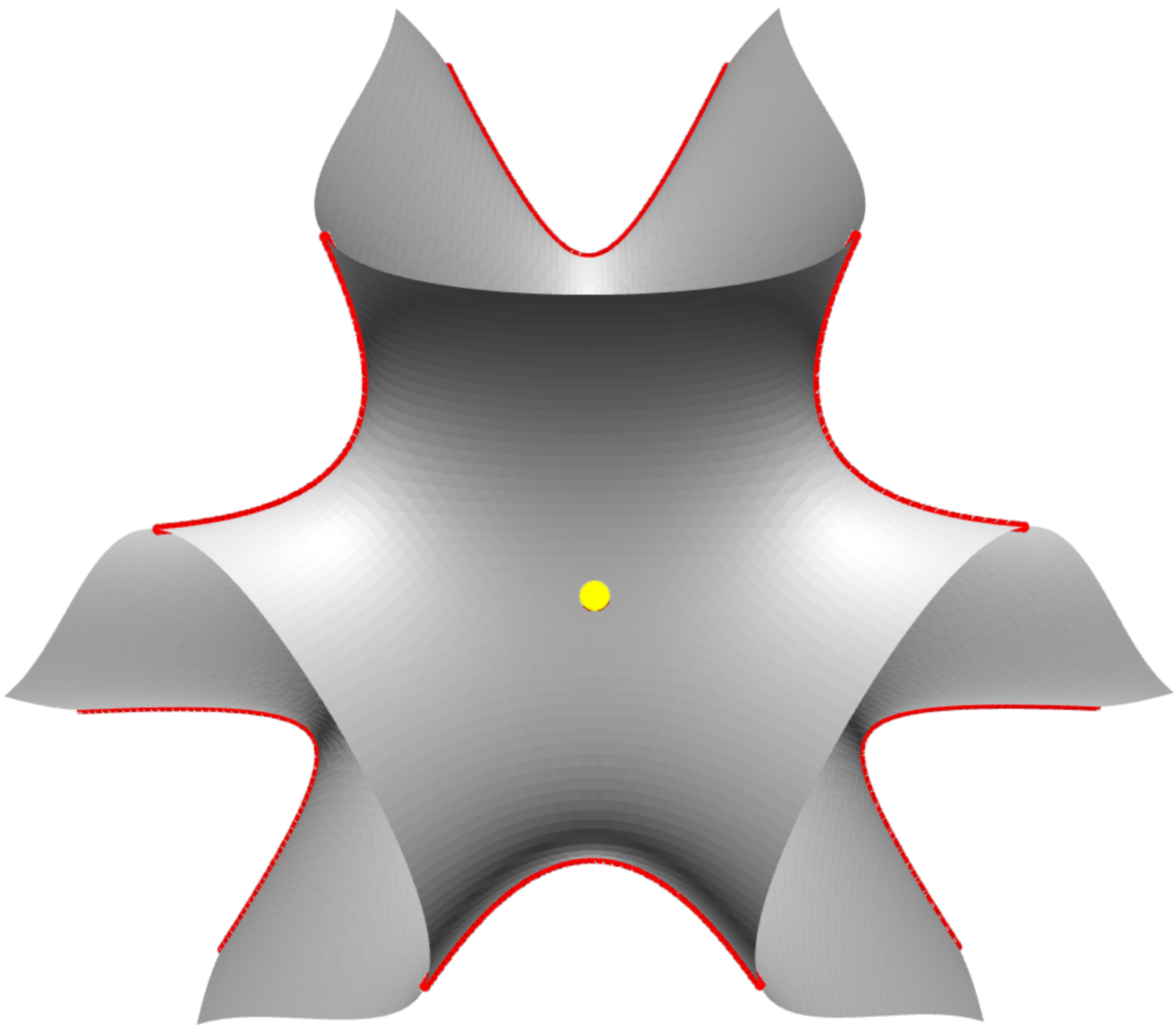
$$\begin{tikzcd} \mathbb{P}^n - \Lambda \arrow[dr] & \\ X \arrow[u, phantom, "\cup"] \arrow[r, "\pi_\Lambda"] & \mathbb{P}^r \end{tikzcd}$$

Finite

$$R(\Lambda) := \text{Ram. div. of } \pi_\Lambda .$$

"Apparent boundary"

Apparent Boundary



PROJECTION-RAMIFICATION MAP

$X^r \subset \mathbb{P}^n$

$$\begin{tikzcd}
\Lambda \arrow[r, mapsto] & R(\Lambda) \\
\mathrm{Gr}(n-r, n+1) \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[d, phantom, "\rotatebox{90}{$=$}"] & {|K_X + (r+1)H|} \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[d, phantom, "\rotatebox{90}{$\cong$}"] \\
\mathrm{Gr} \arrow[r, dashed, "\rho"] & \mathbb{P}^N
\end{tikzcd}$$

QUESTIONS

Is ρ injective / surjective ?

Local version:

$$\mathrm{Def}_\Lambda \longrightarrow \mathrm{Def}_{R(\Lambda)}$$

For a general Λ is ρ injective / surjective on Def ?

Equivalently is ρ generically finite / dominant ?

↳ "maximal variation" of $R(\Lambda)$

HISTORY

We observed

$$\mathrm{Map}(\mathbb{P}^1, \mathbb{P}^n; d) \dashrightarrow \mathrm{Map}(\mathbb{P}^1, \mathbb{P}^{n-1}, d')$$

that turned out to be ρ for scrolls

Flenner, Manaresi, Ciliberto, Zak ...
Studied maximal variation to
Understand the Stückrad-Vogel
cycle.

Dimension Count

$$Gr(n-r, n+1) \overset{\rho}{\dashrightarrow} |K_X + (r+1)H|$$

$\dim = (n-r)(r+1)$

?

Proposition:

$$(n-r)(r+1) \leq \dim |K_X + (r+1)H|$$

with equality if and only if

$$\deg X = n - r + 1$$

(X is of minimal degree)

EXPECTATION

$$\mathrm{Gr} \overset{\rho}{\dashrightarrow} \mathbb{P}^N$$

Should be

generically finite always

& also dominant if X

is of min deg.

(Almost, but not exactly.)

DOMAIN OF DEFINITION

$$\rho : \Lambda \longmapsto R(\Lambda)$$

NO PROBLEM IF $\Lambda \cap X = \emptyset$

OR EVEN IF

$$\pi_\Lambda : X \dashrightarrow \mathbb{P}^r$$

IS DOMINANT.

DEFINITION: $X \subset \mathbb{P}^n$ INCOMPRESSIBLE if $\pi_\Lambda : X \dashrightarrow \mathbb{P}^r$ dominant for all Λ

$X \subset \mathbb{P}^n$ INCOMPRESSIBLE

$\Rightarrow \quad \rho : G_r \longrightarrow \mathbb{P}^N$ regular

$\Rightarrow \quad \rho$ finite.

Examples

1. Curves
2. Hypersurfaces

Rare

DUAL

$$X \subset \mathbb{P}^n = \mathbb{P}V$$

$$X^* \subset \mathbb{P}V^*$$

$$\parallel$$

$$\{H \mid X \cap H \text{ is singular}\}$$

THEOREM

If $X^* \subset \mathbb{P}V^*$ is a hypersurface, then ρ is generically finite for X.

"Non-degenerate dual"

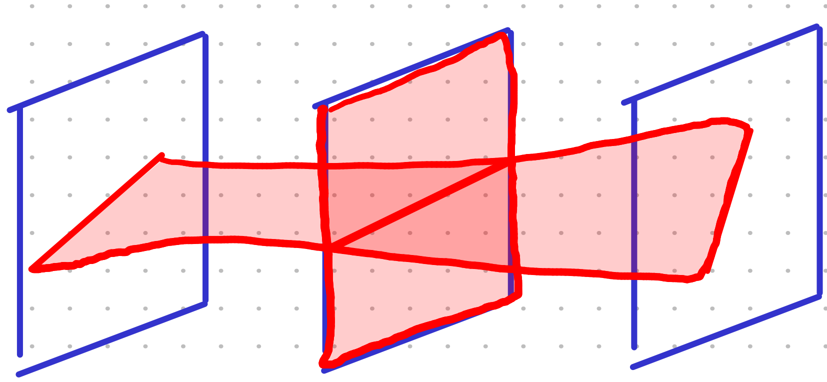
NON-DEGENERATE DUAL

Ubiquitous

$\Leftrightarrow$ For a general H such that $X \cap H$ is singular, it is singular at finitely many points.

Non-Example: $r \geq 3$

$X = \mathbb{P}^1 \times \mathbb{P}^{r-1} \hookrightarrow \mathbb{P}^{2r-1}$ (Segre)



More generally $X = \mathbb{P}E \xhookrightarrow{\mathcal{O}(1)} \mathbb{P}^n$

ρ FOR VARIETIES OF MIN DEG

Generically finite $\Leftrightarrow$ Dominant

1. Rational normal curves ✓
2. Quadric hypersurfaces ✓
3. Veronese $\mathbb{P}^2 \subset \mathbb{P}^5$ ✓
4. Scrolls ?

$\hookrightarrow$ Subtle!

ρ FOR SCROLLS

EXAMPLE OF FAILURE

$$X = \mathbb{P}\left(\mathcal{O}(1)^{r-1} \oplus \mathcal{O}(2)\right) \qquad r \geq 4$$

$$\mathrm{Gr}(n-r, n+1) \overset{\rho}{\dashrightarrow} |K_X + (r+1)H|$$

$\circlearrowleft$ Aut X $\qquad\qquad$ $\circlearrowleft$ Aut X

No gen stab $\qquad\qquad$ YES gen stab (Pos. dim.)

$\Rightarrow$ ρ can't be generically finite.

ρ FOR SCROLLS

THEOREM: Fix r.

Let E be the generic vector bundle on $\mathbb{P}^1$ of rank r and degree

$$d \geq (r-1)(2r-1)+1.$$

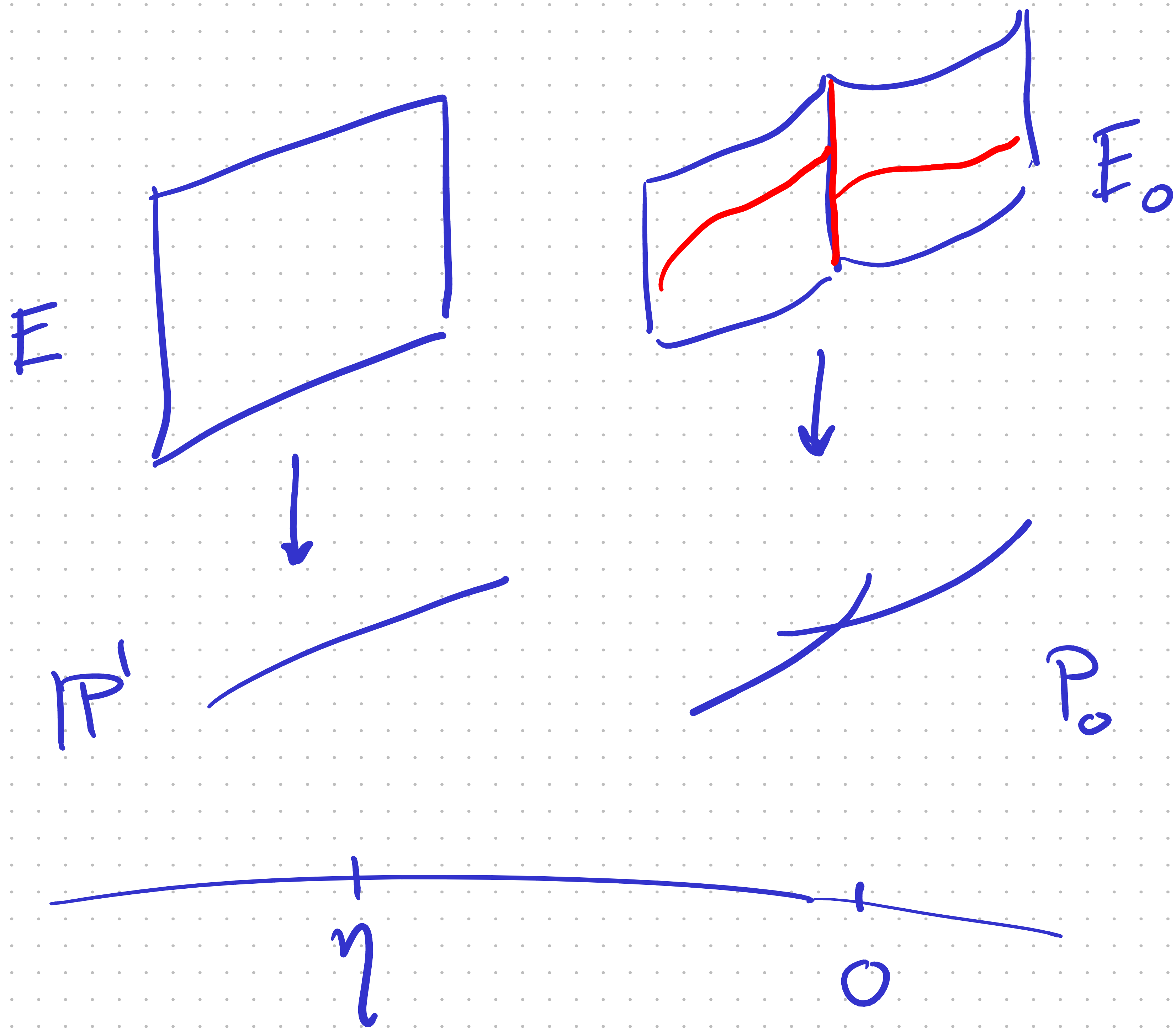
Then ρ is dominant for

$X = \mathbb{P}E$.

Generic scrolls of high degree ✓

SKETCH OF PROOF

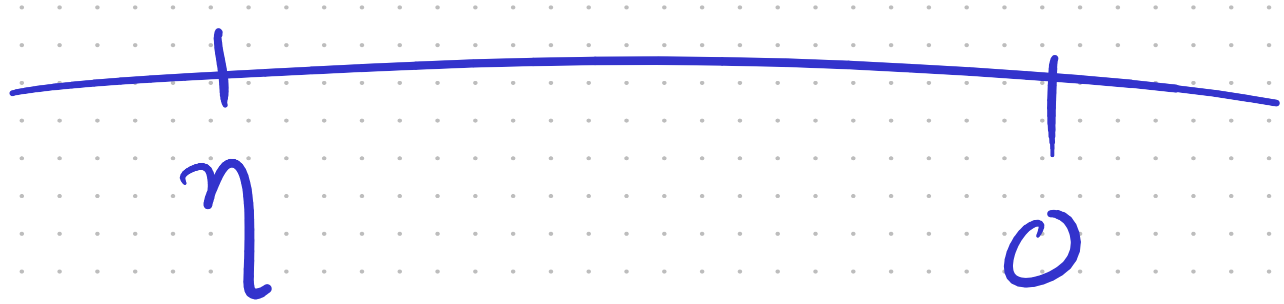
Degenerate.



SKETCH OF PROOF

$$\begin{tikzcd}
Gr(n-r, H^0(X,\mathcal{O}(1))) \arrow[d, equal] \\
Gr(n-r, H^0(E)) \arrow[d, dashed, "\rho"] \\
Gr(1, H^0(E \otimes \det E \otimes \omega_{\mathbb{P}^1}))
\end{tikzcd}
\qquad
\begin{tikzcd}
Gr(n-r, H^0(X_0,\mathcal{O}(1))) \\
Gr(n-r, H^0(E_0)) \arrow[d, dashed, "\rho_0"] \\
Gr(1, \cdots\cdots)
\end{tikzcd}$$

Never dominant
$R(\Lambda)$ contains fiber over node.



SKETCH OF PROOF

Rescue : Limit linear series

Previously : $$\begin{array}{c} Gr(\pi_* \mathcal{E}) \\ \downarrow \\ \Delta \end{array}$$

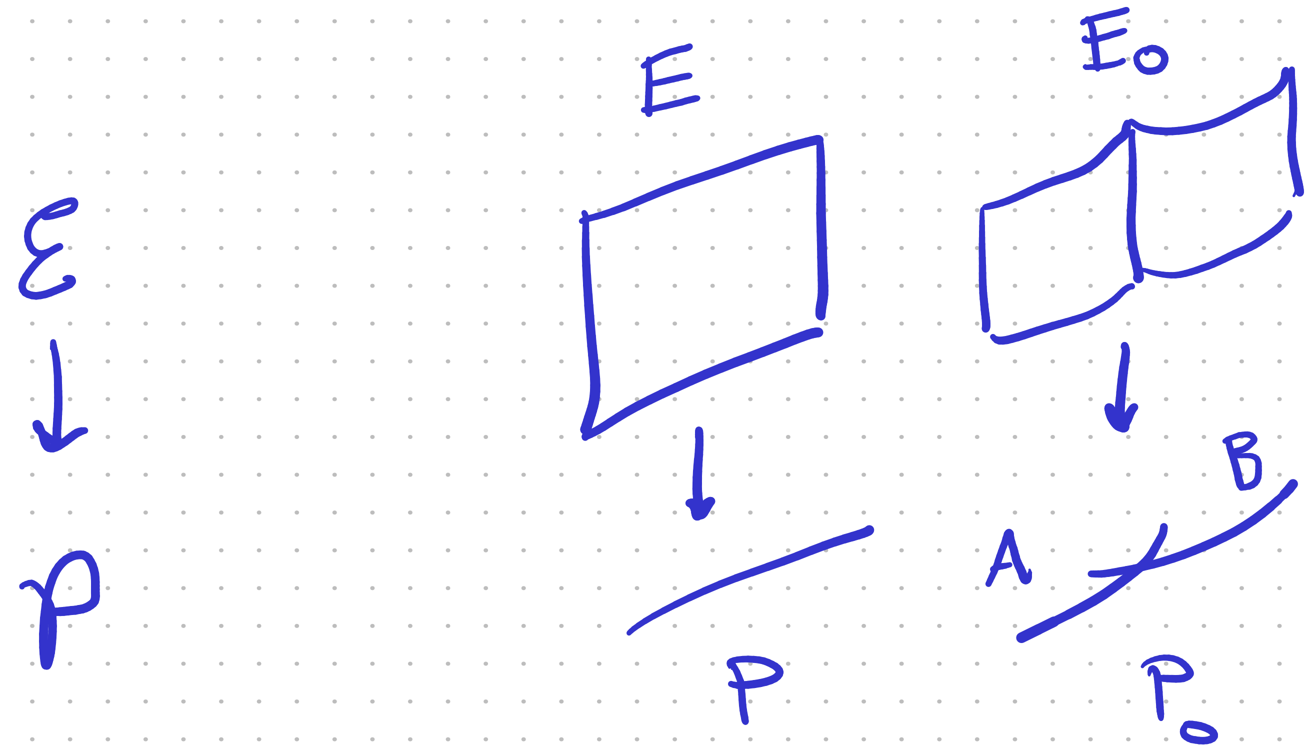
New : $$\begin{array}{c} \mathcal{G}(\pi_* \mathcal{E}) \\ \downarrow \\ \Delta \end{array}$$

Constructed by

Eisenbud - Harris (rank 1)

Teixidor - i - Bigas & Osserman

IDEA OF LIMIT LINEAR SERIES



Can change $\mathcal{E}$ to $\mathcal{E} \otimes \mathcal{O}(nA) =: \mathcal{E}_n$

A subspace of $H^0(\mathcal{E}_\eta)$ has limits in $H^0(\mathcal{E}_n|_0)$ for all n

A limit linear series of rank k is a collection of rank k subspaces of $H^0(\mathcal{E}_n|_0)$ for all $n \in \mathbb{Z}$ satisfying certain conditions.

ENUMERATIVE PROBLEMS

For every $X \subset \mathbb{P}^n$ of minimal degree, find the degree of ρ.

ENUMERATIVE PROBLEMS

① $X \subset \mathbb{P}^n$ rational normal curve.

$$Gr(n-2, n) \xrightarrow{\rho} \mathbb{P}^{2n-2}$$

Regular & $\rho^* \mathcal{O}(1) = \mathcal{O}(1)$

So $\deg \rho = c_1(\mathcal{O}(1))^{top}$

$$= \frac{(2n-2)!}{n!\,(n-1)!}$$

ENUMERATIVE PROBLEMS

(2) $X \subset \mathbb{P}^n$ quadric hypersurface

$\mathbb{P}^n = \mathbb{P}V$

Then

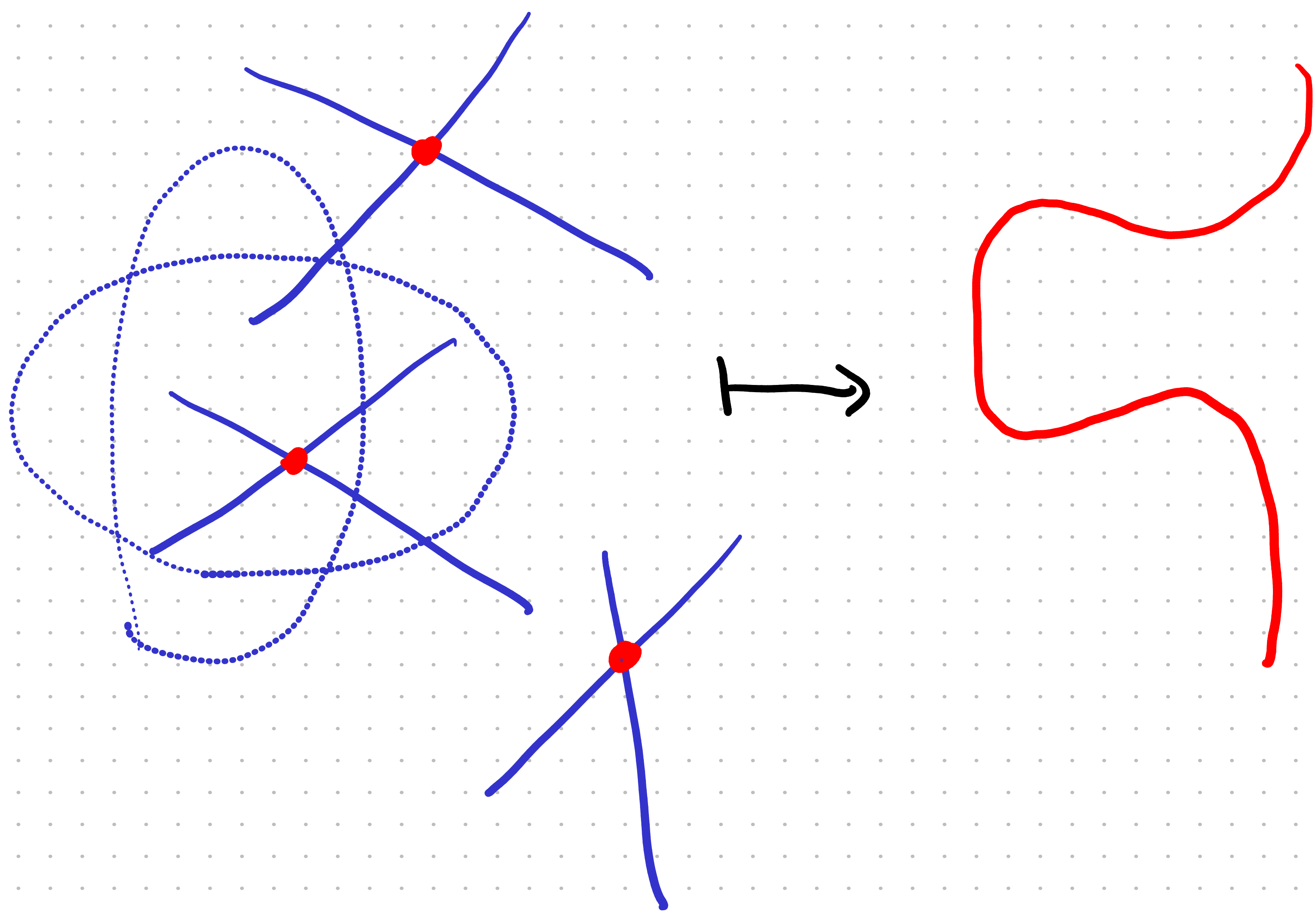
$$\begin{tikzcd} Gr(1,V) \arrow[r, "\rho"] \arrow[d, equal] & \mathbb{P}|K_X + nH| \arrow[d, equal] \\ \mathbb{P}V \arrow[r] & \mathbb{P}V^* \end{tikzcd}$$

Isomorphism induced by X

ENUMERATIVE PROBLEMS

③ $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$

$$\left\{\begin{matrix}\text{Net of} \\ \text{Conics}\end{matrix}\right\} \overset{\rho}{\dashrightarrow} \{\text{cubic}\}$$



Has degree 3.

ENUMERATIVE PROBLEMS

Scrolls : ?

dim \ deg									d
1	1	2	5	14	42	132	429	. . .	Catalan
2		1	1	2	6	22	92	422 . . .	A001181
3			?	?	?				
4			?	?	?				
⋮									
r									

Degree of P for generic Scroll of dim r deg d.