Points to get across –

Topology	⟷	Algebra
Description of points.		Description of points
× monodromy		× Casnati theorems
× connectedness		× Unirationality q.
× Smoothness		× V.b. theorem.
		× Large d regime.

Cohomology	
× Euler char.	× Picard groups conj.
× Conjectures.	× Picard gp thm.
× Simplicial complex	× Large d regime.

Stable cohomology/chow – Ongoing work.

Key ideas : View it as space of maps

× Abr. Vist. Compactification

× My compactification

× EV motivation

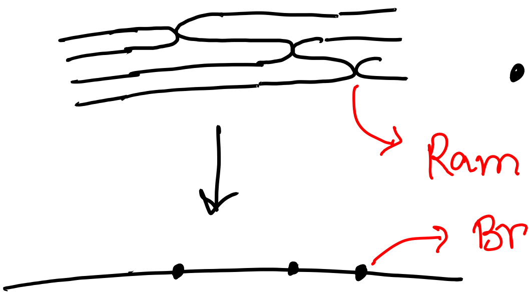
× Mori motivation.

Geometry of Hurwitz Spaces

B a compact Riemann surface of genus $h \geq 0$.
$g \geq 0$, $d > 0$ integers.

$$H_g^d(B) = \{ f : C \to B \mid \begin{array}{l} C \text{ a smooth R.S. of genus } g \\ f \text{ simply branched of deg } d \end{array} \}$$

Simply branced = • All ramification pts are simple (locally $f : z \mapsto z^2$)
• Images of ram. pts. are distinct.



$$\# \text{Ram} = \# \text{Br} = \underbrace{(2g-2) - d(2h-2)}_{b}$$

$$H_g^d := H_g^d(\mathbb{P}^1) / \text{Aut}(\mathbb{P}^1).$$

$H_g^d(B)$ = smooth q. proj $\mathbb{C}$-var. of dim b.

Topology — $\underline{\underline{\text{Both}}}$ — algebra / alg. geo.

$$H_g^d(B)$$

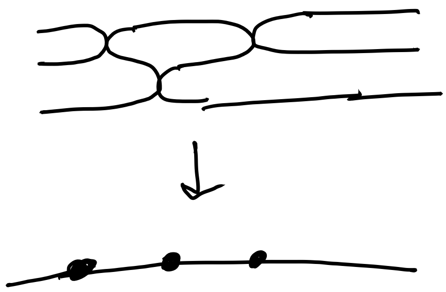
(I) Explicit description of Points

Topology

Covering space of deg d.

$$\begin{tikzcd} C^\circ \arrow[r, hook] \arrow[d] & C \arrow[d, "f"] \\ B^\circ \arrow[r, hook] & B \end{tikzcd}$$

$B^\circ = B \setminus br(f)$



$$\{C^\circ \to B^\circ\} \longleftrightarrow \{\varphi : \pi_1(B^\circ) \to S_d\} / \text{conj.}$$

① $\varphi : \circlearrowleft \longmapsto (ij)$

② $\operatorname{Im} \varphi \subset S_d$ is transitive.

Conversely any such φ gives $f : C^\circ \to B^\circ$ which uniquely extends to $f : C \to B$.

So $\{f: C \to B\} \longleftrightarrow \{$ b distinct pts on B
$+ \ \varphi: \pi_1(B^\circ) \to S_d \,/\text{conj}\}.$

$$\begin{array}{ccc} H_g^d(C) & \leftarrow & \text{Fiber} \cong \{\varphi: \pi_1(B^\circ) \to S_d\}/\text{Conj} \\ \downarrow & & \\ (\mathrm{Sym}^b(B) \setminus \mathrm{Disc}) & & \end{array}$$

covering space

finite

indep of b pts.

Algebra

$\deg f = d$

$$\begin{array}{c} C \\ f \downarrow \\ B \end{array}$$

- $d=2$. Zariski Locally
  $C = \mathrm{Spec}\ \mathcal{O}_B[y]/(y^2-f).$

  Globally
  $C = \mathrm{Spec}(\mathcal{O}_B \oplus \mathcal{L})$ where
  mult defined by $f: \mathcal{L}^2 \to \mathcal{O}.$

- In general
  $C = \mathrm{spec}(A)$ where A is an $\mathcal{O}_B$ algebra, locally free of rank d as an $\mathcal{O}_B$-mod.

$$\{f: C \to B\} \longleftrightarrow \left\{ \begin{array}{c} \mathcal{O}_B\text{-algebra } \mathcal{A} \\ \cdots\cdots \end{array} \right\}$$

Closer look at $\mathcal{A}$

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_B \arrow[r] & \mathcal{A} \arrow[r] \arrow[l, bend left, "\frac{1}{d}\text{trace}"] & F \arrow[r] & 0 \end{tikzcd}$$

So $\mathcal{A} = \mathcal{O}_B \oplus F$

$\hookrightarrow$ Locally free of rank $(d-1)$ degree $-(g-1)+d(h-1)$.

$$\begin{tikzcd} \{f: C \to B\} \arrow[d, "T"] & \text{Fiber} \cong \{\text{Alg. structure on } \mathcal{O}_B \oplus F\} \arrow[l] \arrow[d] \\ \{\text{V.b. of rank } (d-1) \text{ on } B\} & \ni F \end{tikzcd}$$

<u>Q</u>: Give an explicit description of the fibers.

① For which F is the fiber non-empty, e.g. generic F?

② dim of fiber over given F?

<u>UNKNOWN</u> - even for $B = \mathbb{P}^1$.

$B = \mathbb{P}^1$ F generic.

① $\tau^{-1}(F) \subset H_g^d(\mathbb{P}^1)$ is non empty open ✓

For $d \leq 5$, we have a good understanding of $\tau^{-1}(F)$.

<u>Thm</u>: (Casnati-Ekedahl) For $d \leq 5$, there is an explicit surjective (algebraic) map

$$\begin{array}{ccc} V & \longrightarrow & U \\ \text{open } \cap & & \cap \text{ open} \\ \mathbb{A}^N & \dashrightarrow & H_g^d(\mathbb{P}^1). \end{array}$$

<u>Open Q</u>: For which d (and g) is there a dominant map

$$\mathbb{A}^N \dashrightarrow H_g^d \quad ?$$

- known for a finite list of (d, g) by Geiss, Schreyer.
- Similar results about fibers of τ on an arbitrary B for $d \leq 5$

No explicit alg. structure thm for $d \geq 6$.

Thm (–, Patel): Let F be a general v.b of rank $(d-1)$ and degree $-(g-1)+d(h-1)$ on B. If $g \gg h$ ($\sim d^3 h$), then $T^{-1}(F)$ is non-empty.

$B = \mathbb{P}^1$. A "top down" description of pts

$$\begin{array}{ccc} C \downarrow f & \leftarrow \text{starting point} & \in \mathcal{M}_g \\ \mathbb{P}^1 & + & \uparrow \\ & \text{Line bundle } L \text{ of deg } d & \in \mathrm{Pic}^d(\mathcal{C}_g/\mathcal{M}_g) \\ & + & \uparrow \\ & \text{2-sections/scaling} & \mathbb{P}(\pi_* \mathcal{L}^{\oplus 2}) \end{array}$$

Useful if $d > 2g-2$.

$$H_{d,g}(\mathbb{P}^1) \underset{\text{open}}{\subset} \mathbb{P}(\pi_* \mathcal{L}^{\oplus 2})$$

$$H_{d,g} \underset{\text{open}}{\subset} Gr(2, \pi_* \mathcal{L})$$

II. Cohomology $(B = \mathbb{P}^1, \ / \mathrm{Aut}(\mathbb{P}^1))$.

$H_g^d \longrightarrow \mathcal{M}_{0,b}$ étale cover.

$\rightsquigarrow$ Euler. char ✓

Q: Find $H^n(H_g^d, \mathbb{Q})$ or

$A^m(H_g^d, \mathbb{Q})$.

in particular for $m=1$ (or $n=2$)

Conj (Franchetta conj). $A^1(H_g^d, \mathbb{Q}) = 0$.

True for $d \leq 5$ (–, Patel)

for $d \geq 2g-2$

$\hookrightarrow$ Eqv. to $A^1(\mathcal{M}_g, \mathbb{Q}) = \mathbb{Q}$ (Harer).

Stable cohomology:

Conj (Ellenberg-Venkatesh-Westerland)

$$\lim_{d\to\infty} H^n(H_g^d, \mathbb{Q}) = 0. \quad \text{for } n \geq 2$$

Madsen-Weiss + Ebert-Randall-Williams + ε

$$\lim_{g\to\infty} \lim_{d\to\infty} H^n(H_g^d, \mathbb{Q}) = 0 \quad \text{for } n \geq 2$$

III. Topology + Algebra — H_g^d as mapping spaces

$$\left\{\begin{array}{c} C \\ \downarrow \\ B \end{array}\right\} \longleftrightarrow \{\varphi : \pi_1(B^\circ) \to S_d\} / \text{conj}.$$

$$\longleftrightarrow \underbrace{\mu : B^\circ \to BS_d.}$$

↪ makes sense in alg.geo!

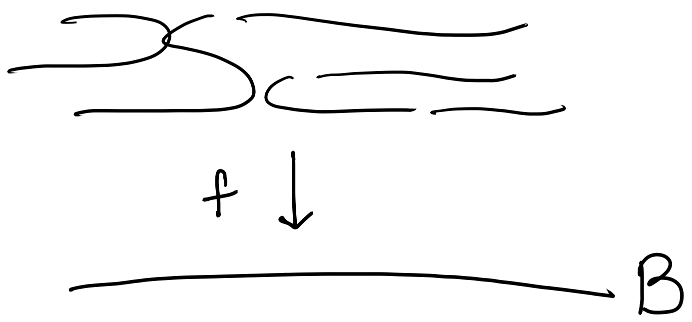
So $H_g^d(B)$ = Space of maps of b-punctured B into $\underline{\underline{BS_d}}$.

↪ DM stack.

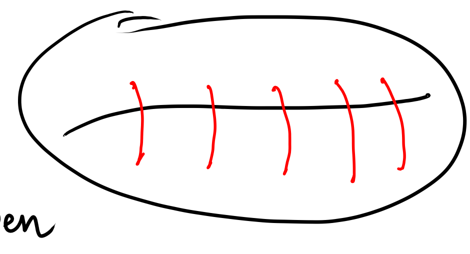
⇒ Compactification of H_g^d (Abramovich–Corti–Vistoli)

⇒ Stable coh. conj. (Ellen. Venk. West.)

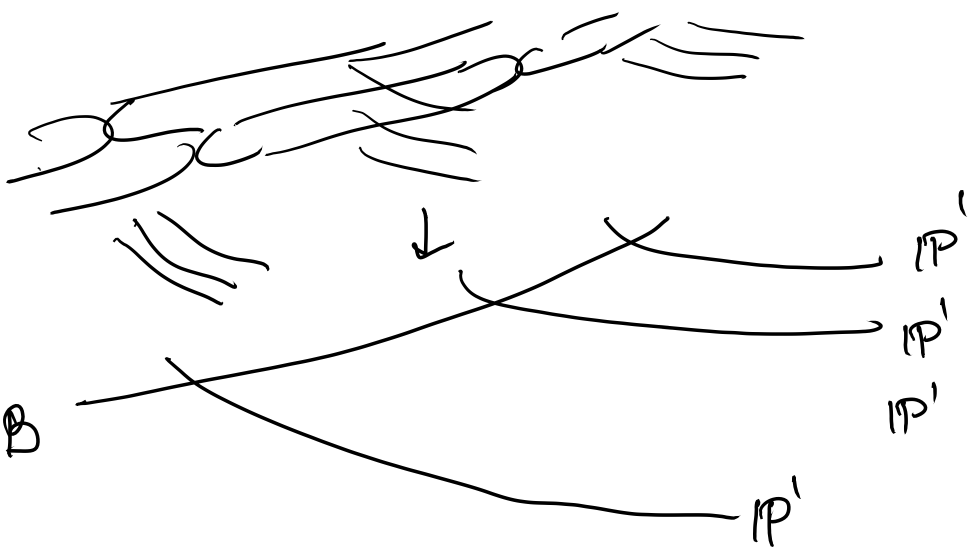
⇒ main idea in pf of first thm.



wanted to deform f.

$\mu : B \longrightarrow$ 

__Mori__ — Attach many $\mathbb{P}^1$s & then the curve deforms.



Then the cover deforms!

• <u>Quasi-modularity</u> $C_g \xrightarrow{d} E$ simply br.

$$\sum N(g,d)\, q^d \quad \text{is a q.m.f.}$$

• <u>ELSV</u> $\alpha = (\alpha_1, \dots, \alpha_m)$ genus g.

$$H^g_\alpha = C(g,\alpha) \int_{\overline{M}_{g,n}} \frac{1 - \lambda_1 + \lambda_2 \cdots \pm \lambda_g}{(1-\alpha_1\psi_1)\cdots(1-\alpha_m\psi_m)}$$

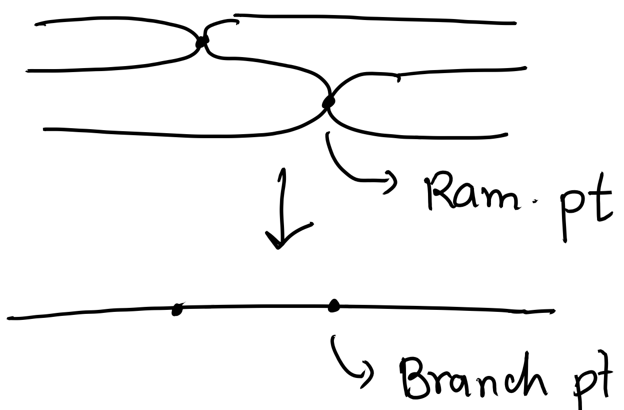
<u>Cor</u>: Polynomial in $\alpha_1, \dots, \alpha_m$.

Geometry of Hurwitz Spaces

B = Compact R.S. of genus $h \geq 0$

$$H^d_g(B) = \left\{ \begin{array}{c} C \\ f \downarrow \\ B \end{array} \;\middle|\; \begin{array}{l} C \text{ cpt R.S. genus } g \\ f \text{ deg } d \text{ simply branched} \end{array} \right\}$$

Simply branched -



- Ram pts of index 1 (Locally $z \mapsto z^2$)
- Ram pts in distinct fibers.

$$\# \text{Ram} = \# \text{Br} = \underbrace{(2g-2) - d(2h-2)}_{b}$$

$$H_g^d(B) = \text{Smooth quasi proj dim } b$$

$$H_g^d := H_g^d(\mathbb{P}^1) / \mathrm{Aut}(\mathbb{P}^1)$$

$$= \text{Smooth quasi proj. dim } b-3.$$

$$\begin{tikzcd}
 & H_g^d(B) \arrow[dl, rightsquigarrow] \arrow[dr, rightsquigarrow] & \\
\text{Topology} & & \text{Algebra.}
\end{tikzcd}$$

$\underline{Q}$: How does one "write down" the points of $H_g^d(B)$?

① Topology –

$$\begin{tikzcd} C \arrow[d, "f"'] & \supset & C^\circ \arrow[d, "f^\circ"] \\ B & \supset & B^\circ \end{tikzcd} \leftarrow \text{cov. sp. deg } d$$

$$B^\circ = B \setminus br(f)$$

$$f^\circ \leftrightarrow \{\mu : \pi_1(B^\circ) \longrightarrow S_d\} / \text{conj}.$$

...

Point of $H_g^d(B) = \begin{tikzcd} C \arrow[d, "f"] \\ B \end{tikzcd}$

$$= \quad B^\circ \subset B + \{\mu : \pi_1(B^\circ) \to S_d\} / \text{Conj}$$

...

compl. of b pts.

$$
\begin{tikzcd}
H_g^d(B) \arrow[d, "\text{finite cov. space.}"] & \{\pi_1(B\setminus\Sigma) \to S_d\}/\sim \arrow[l] \arrow[d] \\
\mathrm{Sym}^b(B)\setminus \mathrm{Disc} & \{\Sigma\} \arrow[l]
\end{tikzcd}
$$

<u>Algebra</u> ("Write down points").

$$
\begin{array}{c} C \\ f \downarrow \\ B \end{array}
\qquad
\begin{array}{l}
\deg f = 2 \\
\text{Locally } C = \{ (y,x) \mid x \in B, \; y^2 - f(x) = 0 \} \\
\text{for some } f \in \mathcal{O}_B.
\end{array}
$$

Line bundle deg $-\frac{b}{2}$

Globally $C = \underline{\mathrm{Spec}}\,(\mathcal{O}_B \oplus \mathcal{L})$

where the algebra structure given by

$f : \mathcal{L}^2 \to \mathcal{O}_B$

Higher degree.

$$C = \underline{\mathrm{Spec}}(\mathcal{O}_B \oplus \mathcal{F})$$

$$\begin{array}{c} C \\ \downarrow \\ B \end{array}$$

$\mathcal{F}$ locally free of rank $(d-1)$ & degree $-\frac{b}{2}$.

$$\left\{\begin{array}{c} C \\ \downarrow \\ B \end{array}\right\} = \text{V.b. } \mathcal{F} + \text{$\mathcal{O}_B$- alg. structure on $\mathcal{F}$}$$

??

<u>Q</u>

- For which F is it possible to give $\mathcal{O}_B + F$ an alg. str ... ?
- How many alg. str ?

$$\begin{array}{ccc} H_g^d(B) & \ni & f \\ \downarrow & & \downarrow \\ \{\text{V.b. on } B\} & \ni & F \end{array}$$

- Image ?
- Fibers ?

Unknown even for $B = \mathbb{P}^1$ (unless $d \leq 5$)

$B = \mathbb{P}^1$

<u>Thm</u> (Miranda, Casnati-Ekedahl)
For $d \leq 5$, there is a (explicit) alg. map

$$\begin{array}{ccc} V & \longrightarrow & U \\ \cap & & \cap \\ \mathbb{A}^N & \dashrightarrow & H_g^d(\mathbb{P}^1) \end{array}$$

Zar. open.

<u>Q</u>: Is H_g^d unirational for $d \geq 6$?

<u>Thm</u> (–, Patel): Let F be general of rank $(d-1)$ & deg $-\frac{b}{2}$. If $b \gg 0$, then $\mathcal{O}_B \oplus F$ admits an alg. str. such that $C = \mathrm{Spec}(\mathcal{O} \oplus F)$ is smooth & simply br.

$$H_g^d(B) \longrightarrow \{\text{V.b. on } B \dots\}$$

is dominant for $g \gg h$

- For all d
- Not constructive.

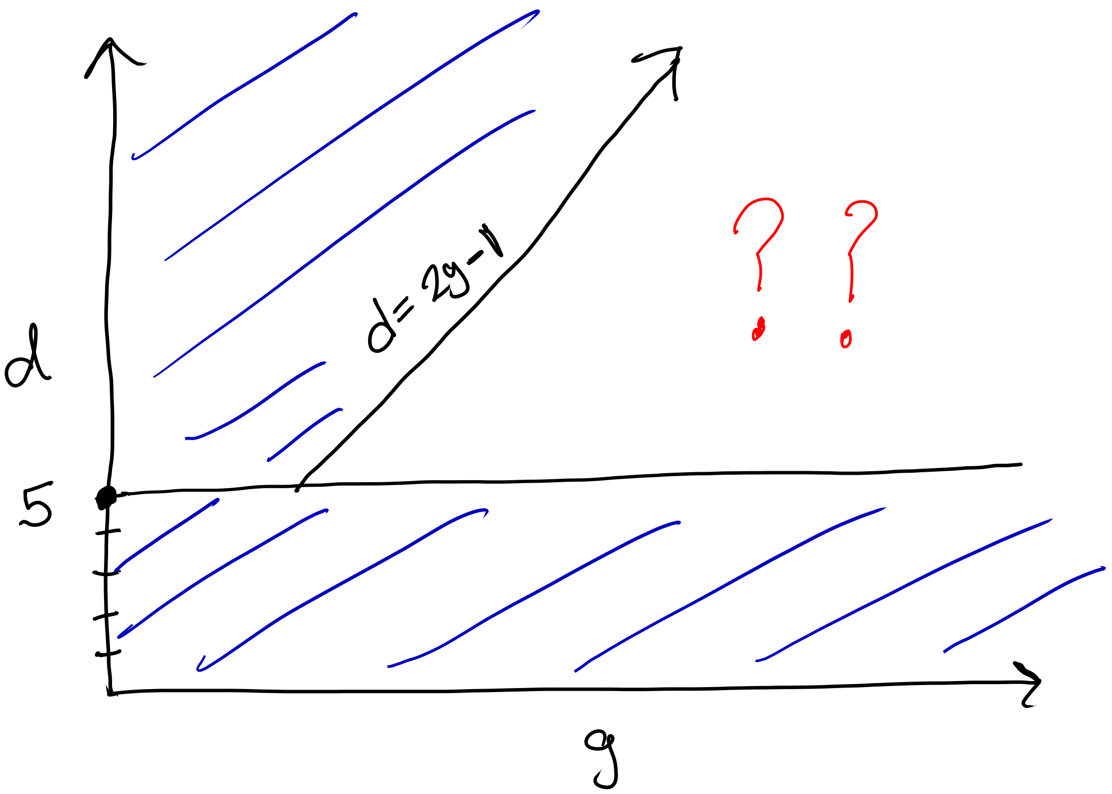
Recall: Algebra to write down pts of $H_g^d(B)$.

"Top down" description $B = \mathbb{P}^1$

$$\begin{array}{ccccc} C & \longleftarrow & \text{curve} & \in & \mathcal{M}_g \\ \downarrow f & & + & & \uparrow \\ \mathbb{P}^1 & & L = f^*\mathcal{O}(1) & \in & \mathrm{Pic}^d(\mathcal{C}_g/\mathcal{M}_g) \\ & & + & & \\ & & 2 \text{ sections / scaling} & \in & \mathbb{P}\left((\pi_*\mathcal{L})^{\oplus 2}\right) \end{array}$$

Useful if $d > 2g-2$.

Picture:- Understanding of H_g^d



Cohomology / Chow ring of H_g^d

$$H_g^d \quad \text{étale of degree} = \text{Hurwitz number.}$$
$$\downarrow$$
$$M_{0,b}$$

$$\chi(H_g^d) = (h_g^d) \cdot \chi(M_{0,b})$$

What about cohomology?

<u>Conj</u> (Franchetta conj)

$$H^2(H_g^d, \mathbb{Q}) = 0$$

<u>Thm</u> for $d \leq 5$ (–, Patel)

for $d > 2g-2$ ($\Leftrightarrow H^2(M_g, \mathbb{Q}) = \mathbb{Q}$ Harer)

Conj (Ellenberg-Venkatesh-Westerlund)

$$\lim_{d\to\infty} H^n(H_g^d, \mathbb{Q}) = 0 \qquad \text{for } n \geq 2$$

Thm - $\lim_{d\to\infty} \lim_{g\to\infty} H^n(H_g^d, \mathbb{Q}) = 0$

(Madsen-Weiss + Ebert-Randal-Williams + ϵ)

A synthesis of top. + alg.

(H_g^d as mapping space

$$\left\{ \begin{matrix} C \\ \downarrow \\ B \end{matrix} \right\} \longleftrightarrow \left\{ B^0 + \mu : \pi_1(B^0) \to S_d \right\}$$

up to $\sim$

$$\longleftrightarrow \{ B^0 + \mu : B^0 \to \underline{\underline{BS_d}} \}$$

DM-Stack

$$H_g^d = \mathrm{Maps}(\mathbf{B}, \mathcal{B}S_d)$$

$\rightsquigarrow$ Kontsevich style compactification

(Abr. Corti. Vist, Harris-Mumford)

Also leads to EVW stabilization conjectures.

Also motivates a key idea to kill Obstructions in def. theory in [D. Patel].