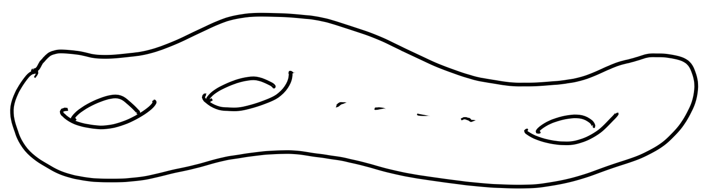
What are ribbons and what do they tell us about Riemann surfaces?

Riemann Surfaces

Riemann surface = Connected compact complex 1-manifold.
= Connected, projective, smooth algebraic curve over $\mathbb{C}$.

Topology :



Captured by genus $g \geq 0$.

But there's much more.

Three perspectives

1. Branched covers

Ex: $f(x) = \sqrt[3]{x^2+1}$ ↙ "multivalued function".

Graph of $f = \{ (x,y) \mid y^3 = x^2+1 \}$.

Gives a Riemann surface X (of genus 1).

Suggests another invariant = # of branches = 3

X also graph of $g(y) = \sqrt[2]{y^3-1}$. # branches = 2.

Gonality of X := Smallest d such that X is the graph of a "d-valued holomorphic function."

More precisely: Smallest d such that $\exists$ surj. holomorphic map

$$X \longrightarrow \mathbb{CP}^1.$$

Gonality $X = 1 \iff X \cong \mathbb{P}^1$

Thm (Segre): Gonality $\leq \left\lceil \frac{\text{genus}}{2} \right\rceil + 1$ and all values from $2, \dots, \lceil \frac{g}{2} \rceil + 1$ are attained.

---

2. Fields.

$$\{\text{Riemann surfaces}\} \;\simeq\; \{\text{fin. gen. fields of tr. deg } 1/\mathbb{C}\}.$$

$$X \longrightarrow K = \text{field of mer. fun. on } X.$$

$$\mathbb{C} \subset \underbrace{\mathbb{C}(t) \subset K}_{\text{degree } d.}$$

$$\mathrm{Gon}(X) = \{\min d \mid K \text{ is a degree } d \text{ ext}^n \text{ of } \mathbb{C}(t)\}$$

3. Projective geometry.

$$X \subset \mathbb{P}^N = \{[X_0 : \cdots : X_N]\}$$

X = Zero locus of homog polys in $X_0, \ldots, X_N$.

$S := \mathbb{C}[X_0, \ldots, X_N]$.

X = Zero locus of a homog. ideal $I \subset S$.

Invariant of I as an S-module

$$I \leftarrow \underbrace{F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_n}_{\text{minimal free resolution}}$$

$$F_i = \bigoplus_j S(-i-j)^{\beta_{ij}} \longleftarrow \text{Invariants of } I.$$

Betti-table of I = (β_{ij}).

Rem: These depend on I, which depends on the map $X \to \mathbb{P}^n$

But there is a canonical map

$X \to \mathbb{P}^{g-1}$ given by differential forms on X.

Henceforth use this.

Conj (Green, rough) :-

Gonality of X $\longleftrightarrow$ Support of betti table of $X \subset \mathbb{P}^{g-1}$.

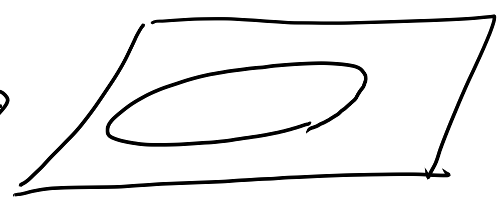
Thm: (Aprodu-Farkas, —) : Green's conj holds for "almost all" curves of every gonality.

(Using Thm of Voisin for curves of max. gonality.)

Ribbons

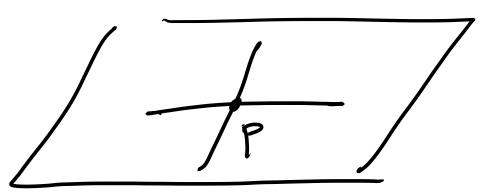
Ex. Take $\mathbb{P}^2 = \{[X:Y:Z]\}$

$XY - Z^2 = 0 \rightsquigarrow$



A smooth curve.

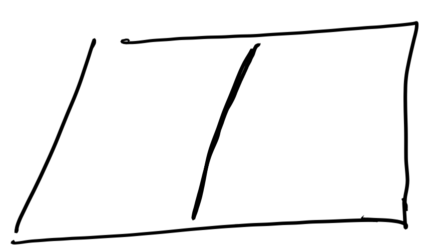
$XY = 0 \rightsquigarrow$



A singular curve.

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0 \text{ at } p.$$

$X^2 = 0 \rightsquigarrow$



geom $\nearrow$ Smooth, same as $X=0$

alg. $\searrow$ singular curve, ↳ everywhere!

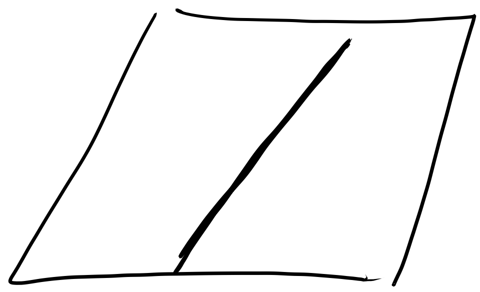
Grothendieck – "Schemes" to formalizes the second choice.

Scheme = top. space X + Sheaf of rings.

"functions on X"

may have nil potents.

$X^2=0 \rightsquigarrow$

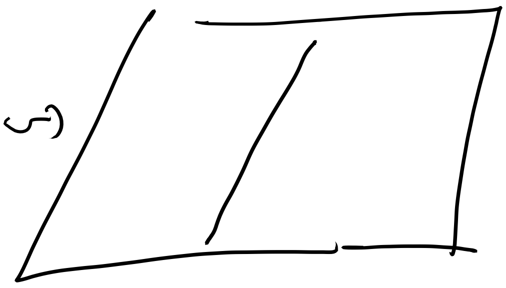


A scheme with nilpotents ("non-reduced") that is singular everywhere

$\neq$

Reduced scheme with no sing. pts.

$X=0 \rightsquigarrow$



Easier example:

$\mathcal{E} = \{\cdot\}$, Ring of functions $\mathbb{C}[x]/x^2$.

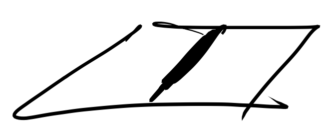
" $\bullet$ "

Fact: Scheme $\{X^2=0\} \subset \mathbb{P}^2$ is locally iso to scheme $\{X=0\} \times \mathcal{E}$

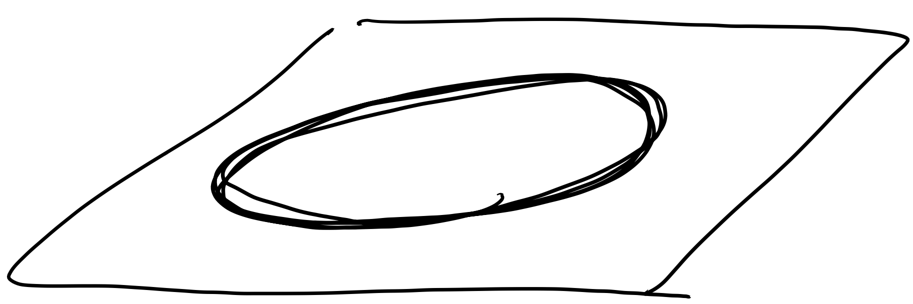
Def: A ribbon is a non-reduced scheme locally isomorphic to smooth curve $\times\ \mathcal{E}$.

Can define genus of a ribbon.

$\text{genus}(X^2=0) = 0.$



$\text{genus}((XY-Z^2)^2) = 3$



$\text{genus}(\mathbb{C}\mathbb{P}^1 \times \mathcal{E}) = -1.$

Can also define "gonality" of a ribbon & make

<u>Conj</u> (Green's conj for ribbons, Bayer-Eisenbud) 1990

Gonality of ribbon $\longleftrightarrow$ Betti table of ribbon.

<u>Thm</u>: (—) Green's conj holds for ribbons. $\square$

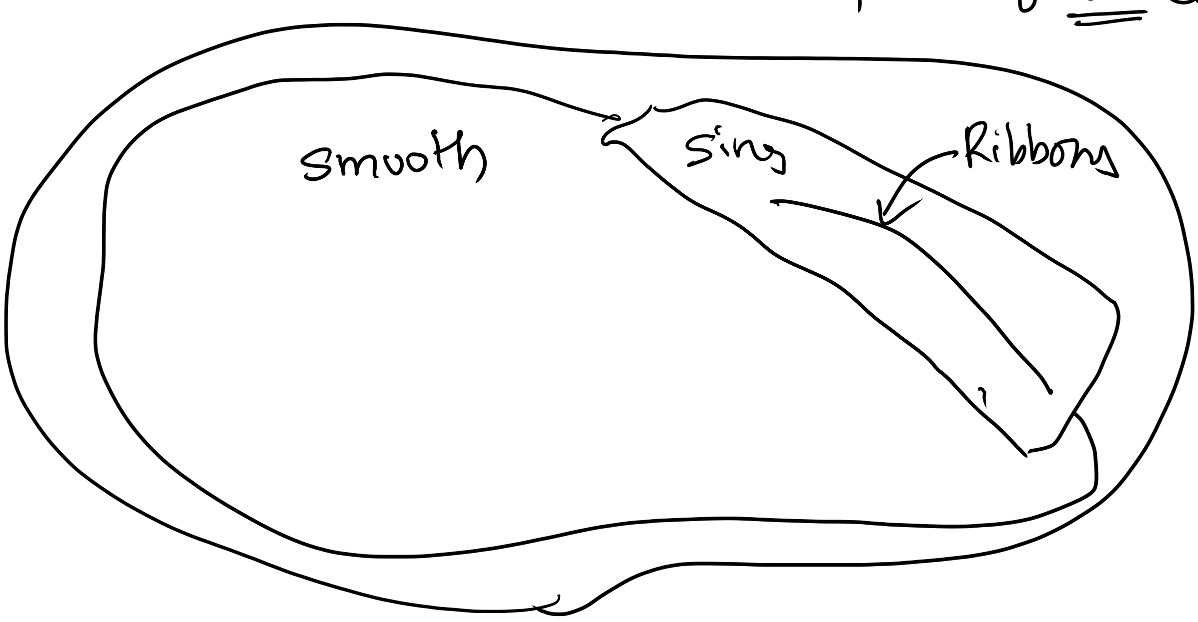
Green's conj for a smooth curve of highest gonality (Voisin) $\xRightarrow{\text{D.}}$ Green's conj for all ribbons of highest gonality

Green's conj for a smooth curve of highest gonality $\Downarrow$ Aprodu-Farkas, Voisin

Green's conj for all ribbons of highest gonality $\Downarrow$ D.

Green's conj for almost all curves of a given gonality. $\xLeftarrow{\text{Easy.}}$ Green's conj for all ribbons.

Space of <u>all</u> curves.

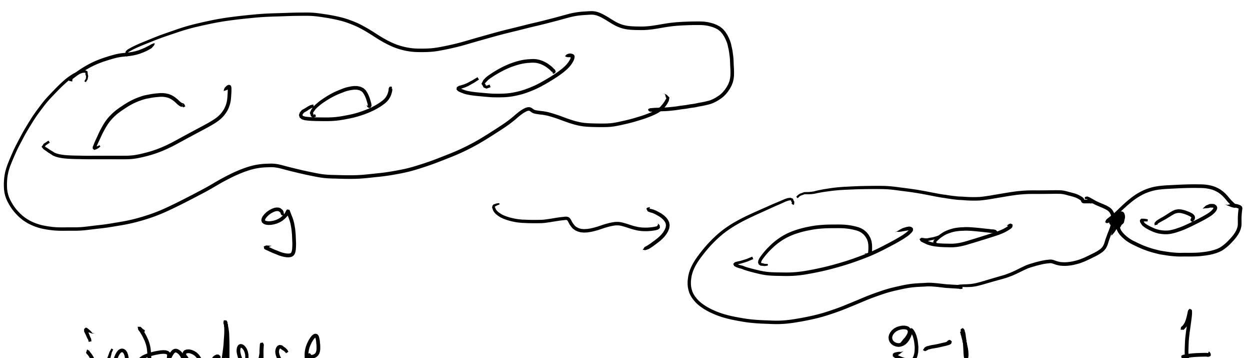


Common feature of many statements in alg. geo. :-

If it holds for one curve, then it holds for "almost all" curves.

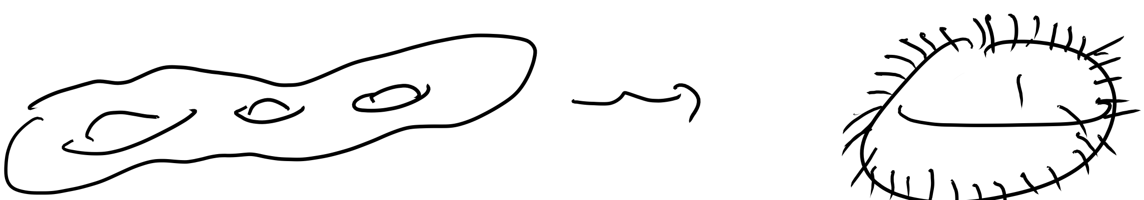
So need to exhibit <u>one</u> curve.

Smooth curves are difficult to write down & analyze.



<u>introduce</u> Singularity $\longrightarrow$ simplifies global geometry
complicates $\underbrace{\text{local geometry}}_{\text{algebra}}$.

<u>Non reduced</u> structures takes this to an <u>extreme</u>.



Geometry of genus g curves $\rightsquigarrow$ algebra of rings attached to $\mathbb{CP}^1$

↑

<u>Sometimes</u> <u>easier</u>!

$\square$

$$x^3 + x^2 y + y^4$$

y

deg 3

deg 4

x

Fix x ask: how many y?

4

$$x^n f_n(y) + x^{n-1} f$$

x-degree

y-degree