What are ribbons and what do they tell us about Riemann surfaces?

**Riemann Surfaces**

Riemann surface = Connected compact complex 1-manifold

= Connected, projective, smooth algebraic curve over \( \mathbb{C} \).

Topology:

![Topology Diagram]

captured by genus \( g \geq 0 \).

But there’s much more.

**Three perspectives**

1. Branched covers

   Ex.: \( f(x) = \sqrt{x^2+1} \leq \text{“multivalued function”} \)

   Graph of \( f = \{ (x,y) \mid y^2 = x^2 + 1 \} \).

   Gives a Riemann surface \( X \) (of genus 1).

   Suggests another invariant = \# of branches = 3

   \( X \) also graph of \( g(y) = \sqrt{y^3 - 1} \).

   \# branches = 2.
Gonality $\gamma_X := \text{Smallest } d \text{ such that } X \text{ is the graph of a } d\text{-valued holomorphic function. }

\text{More precisely: Smallest } d \text{ such that } \exists \text{ surj. holomorphic map } X \to \mathbb{CP}^1

\text{Gonality } X = 1 \iff X \cong \mathbb{P}^1

\text{Thm (Segre): } \text{Gonality } \leq \left\lfloor \frac{\text{genus} X}{2} \right\rfloor + 1

\text{and all values from } 2, \ldots, \left\lfloor \frac{3}{2} \right\rfloor + 1 \text{ are attained.}

\section{2. Fields}

\{ \text{Riemann surfaces} \} \sim \{ \text{fin. gen. fields of tr. deg } 1/\mathbb{C} \}

X \rightarrow K = \text{field of mer. fun. on } X

\mathbb{C} \subset \mathbb{C}(t) \subset K

\text{Gen}(X) = \{ \text{min } d \mid K \text{ is a degree } d \text{ ext. of } \mathbb{C}(t) \}$
3. Projective geometry.

\[ X \subset \mathbb{P}^N = \{ [x_0 : \ldots : x_N] \} \]

\[ X = \text{zero locus of homog. pols in } x_0, \ldots, x_N. \]

\[ S = \mathbb{C}[x_0, \ldots, x_N]. \]

\[ X = \text{zero locus of a homog. ideal } I \subset S. \]

Invariant of \( I \) as an \( S \)-module:

\[ I \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_n \]

minimal free resolution.

\[ F_i = \bigoplus_j S(-i-j) \quad \text{Invariants of } I. \]

Betti table of \( I = (\beta_{i,j}). \)

Remark: These depend on \( I \), which depends on the map \( X \to \mathbb{P}^n \).

But there is a canonical map

\[ X \to \mathbb{P}^{g-1} \]

given by differential forms on \( X \).

Henceforth use this.

Conj (Green, rough):

Gonality of \( X \) <-> Support of betti table of \( X \subset \mathbb{P}^{g-1}. \)
Thm: (Aprodu-Tanaka, —) : Green's conj holds for "almost all" curves of every gonality. (Using Thm of Voisin for curves of max. gonality.)

**Ribbons**

Ex. Take $|P^2| = 3 \{[x:y:z] \}$

$XY - Z^2 = 0 \leadsto \quad \text{A smooth curve.}$

$XY = 0 \leadsto \quad \text{A singular curve.}$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0 \text{ at } p.$$  

$X = 0 \quad \text{geom} \rightarrow \text{Smooth, same as } X = 0$

$\text{alg} \rightarrow \text{singular curve, i.e. everywhere!}$

Grothendieck - "Schemes" to formalize the second choice.

Scheme = top. space + sheaf of rings.

\[ \text{"functions on } X\text{"} \]

may have nilpotents.
A scheme with nilpotents ("non-reduced") that is singular everywhere.

Reduced scheme with no sing. pts.

Easier example:

\[ \mathcal{E} = \{ \cdot 3 \} \text{, Ring of functions } \mathbb{C}[X]/x^2. \]

Fact: Scheme \( \{ X^2 = 0 \} \subset \mathbb{P}^2 \) is locally iso to scheme \( \{ X = 0 \} \times \mathcal{E} \).

Def: A ribbon is a non-reduced scheme locally isomorphic to smooth curve \( \times \mathcal{E} \).

Can define genus of a ribbon.

\[ \text{genus} (X^2 = 0) = 0. \]

\[ \text{genus} ( (XY - Z^2)^2 ) = 3. \]

\[ \text{genus} (CP^1 \times \mathcal{E}) = -1. \]
Can also define "gonality" of a ribbon to make

**Conj** (Green's conj for ribbons, Bayer-Eisenbud 1990)

Gonality of ribbon $\leftrightarrow$ Betti table of ribbon.

**Thm** : (-) Green's conj holds for ribbons.

Green's conj for a smooth curve of highest gonality (Voisin)

\[ \Downarrow \]

Aprodu-Farkas-Voisin.

Green's conj for almost all curves of a given gonality.

Space of all curves

\[ \begin{align*}
\text{Smooth} & \quad \text{Sing} \\
& \quad \text{Ribbons}
\end{align*} \]
Common feature of many statements in alg. geo.:

If it holds for one curve, then it holds for "almost all" curves.

So need to exhibit one curve.

Smooth curves are difficult to write down & analyze.

Introduce singularity \rightarrow simplifies global geometry
complicates local geometry algebra.

Non reduced structures takes this to an extreme.

Geometry of genus \rightarrow algebra of rings attached
up to CP^1

Sometimes easier!
\[ x^3 + x^2 y + y^4 \]

\[ \deg 3 \]

\[ \deg 4 \]

\[ x \]

Fix \( x \) ask: how many \( y \)?

\[ f \in \mathbb{R}[y] \]

\[ \sum_n \alpha_n f_n(y) + x^4 f \]

\( x \)-degree

\( y \)-degree