Log surfaces of almost K3 type and curves of genus 4

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Broader Context -
Problem - Understand compact moduli of varieties of (log) general type.

Fix $\epsilon \in \mathbb{Q}_{>0}$

$\{\text{Smooth} (X,D) \} \cup \{ \text{Degenerate} (X,D) \}$

$\{ k_x + \epsilon D \text{ ample} \}$
Curves:
\[
\{ \text{Smooth } (X,D) \} \cup \{ \text{ } K_X + \varepsilon D \text{ ample} \}
\]
\[
\{ \text{ } \}
\]
\[
\{ \text{ } \}
\]

\(\Rightarrow\) Projective coarse moduli well-understood.

Deligne, Mumford, Knudsen, Harst"ed

Higher dim:
\[
\{ \text{ } (X,D) \text{ sm.} \} \cup \{ \text{ } (X,D) \text{ ...} \}
\]

1. \((X, \varepsilon D)\) has Semi-log-canonical (slc) singularities
2. \(K_X + \varepsilon D\) ample

\(\Rightarrow\) Projective coarse moduli
(CKSB)

Kollár-Shepherd-Barron
Alexeev
Birkar-Cascini-Hacon-McKernan
Xu, Kovács-Patakfalvi

...
• Do not know much about the geometry (sing., tangent spaces, boundary comp...) 
• Probably hopeless - satisfy Murphy's Law.

BUT. Important special cases are better behaved.

Ex. $K_X \sim 0$

• $X$ an Abelian Variety 
  $D \subset X$ the theta divisor.

• $X$ a K3 surfaces of deg 2
  $D \in |L|$ Laza 2012
  $D = \text{Ram.-div. of } X \to \mathbb{P}^2$
  (Alexeev-Engel-Thompson 2018).
Hacking: (2004). Fix a positive integer $d$. KSBA compact $\{(S, (\frac{3}{d}+\epsilon)D)\}$ where $S \cong \mathbb{P}^2$, DCS curve $\gamma$ deg $d$, $\epsilon$ very small.

- Smooth DM stack (if $3 | d$).
- Fairly explicit description of the boundary
- $d=4$ recovers Schubert's compactification of $M_3$.

Salient feature -

$(S, (\frac{3}{d}+\epsilon)D)$ is log gen. type $K_S + (\frac{3}{d}+\epsilon)D$ ample

but barely so.

"Almost K3!"
Main def: Fix a positive rational $r = \frac{m}{n}$

An almost K3 stable log surface is a pair $(S, D)$

- $S$: connected, reduced, Coh. Mau., Proj.
- $D$: Effective Weil divisor on $S$

such that

1. For all sufficiently small $\epsilon > 0$,
   $(S, (r+\epsilon)D)$ is stable.
   $L, SLC + K_S + (r+\epsilon)D$ ample.

2. $nK_S + mD \sim 0$

Remark

1. Both $K_S$ & $D$ are $\mathbb{Q}$-Cartier
2. $S$ smooth $\Rightarrow$ $S$ del Pezzo

Goal: Understand moduli of almost K3 log surfaces.

Today: A highly interesting special case

$S \cong \mathbb{P}^1 \times \mathbb{P}^1$ (generically).

$D \subset S$ of type $(3,3)$. 
Aside: Families

\[ \pi: S \to B \text{ flat proper Cohen-Macaulay} \]
\[ \text{rel dim } 2 \]
\[ DCS \text{ a relative Weil divisor} \]

such that

(i) \( W_{\pi}^{[i]} \) and \( O(D)^{[i]} \) commute with base change \( \forall i \in \mathbb{Z} \).

(ii) All geometric fibers are almost K3 stable log surfaces.

Main theorem:

Let \( \mathcal{X} \) be the moduli stack of smoothable almost K3 stable log surfaces \((S, D)\) with \( K_S^2 = 8 \) & \( p_a(D) = 4 \). Then \( \mathcal{X} \) is proper with proj coarse space.

1. irreducible
2. smooth.

3. the boundary \( (:= \mathcal{X} - \mathcal{X}^0) \)

\[ \mathcal{X}^0 = \{ (S, D) \mid S \text{ & } D \text{ smooth} \} \]

is the union of 4 irreducible divisors.
\( \mathcal{X}^o \rightarrow M_4 \) is an isomorphism onto the complement of the Gieseker-Petri locus. 

\( \mathcal{X}^c \rightarrow M_4 \) is the blow up of the hyperell locus.

\[ \{(s,D) \mid D \text{ smooth}\} \]

\underline{Interior:} \quad \mathcal{X}^o \sim M_4

\( D \) smooth non-hyp. of genus 4

\( D \hookrightarrow \mathbb{P}^3 \) canonical

\( S \) unique quadric (smooth if \( D \) is Gieseker-Petri general).

So \( D \sim (s,D) \)
Boundary:

1. $(S,D) \quad S \cong \mathbb{P}^1 \times \mathbb{P}^1 \quad D \text{ singular \& type } (3,3)$

2. $S \cong \text{Quadric cone, } \text{ DCS general. } \quad \text{in } \left| -\frac{3}{2} K_S \right|$

3. $S = \text{Smoothing of } \mathbb{P}(1,2,9) \text{ at the } \text{A}_r \text{-sing.} \quad \text{DCS general. } \quad \Rightarrow D \text{ hyperelliptic} \quad \text{in } \left| -\frac{3}{2} K_S \right|$

4. $S = \text{Toric deg. of } \mathbb{P}^1 \times \mathbb{P}^1 \text{ given by } \quad \text{D } \in \left| -\frac{3}{2} K_S \right| \quad \text{generic.}$
Proof: (Dream): Understand all $\alpha$-Gor. degenerations of $\mathbb{P} \times \mathbb{P}$ & use this to get the boundary (Manetti, Hacking-Prokhorov for $\mathbb{P}^2$).

(Reality): Tour de force.

Relationship with other moduli spaces

1. $X \rightarrow M_4$

2. $K3$ surfaces.

$(S, D) \mapsto T = \text{cyclic triple cov.}$

of $S$ along $D$.

Lattice Pol.

Hodge Struct. \sim K3 with a lattice polarization coming from the pull-backs of the two rulings $S$

with $M_3$-symm.

$\text{Type I period domain } D$
\[
\begin{align*}
\mathcal{C} & \quad \xrightarrow{\text{(Kondo)}} \quad \mathcal{D}/\Gamma \quad \text{hirvial} \\
\mathcal{M}_4 & \quad \xleftarrow{\mathcal{D}/\Gamma} \quad \text{b.b.} \\
\overline{H}_{3,4}(\frac{1}{6}+\varepsilon) & \quad \xrightarrow{\text{Key to our understanding}} \quad \mathcal{C} \\
\left\{ \begin{array}{l}
\mathcal{C} \text{ veno4} \\
\downarrow 3 \\
\mathbb{P}^1
\end{array} \right\} & = \overline{H}_{3,4} \quad 2:1 \quad \xrightarrow{\text{2:1}} \quad \mathcal{M}_4
\end{align*}
\]