Vector bundles and finite covers

\[ f : X \rightarrow Y \]  
Finite flat \[ f_\ast \mathcal{O}_X \]
Vector bundle on \( Y \).

Question: Which vector bundles arise in this way?

Endow \( V \) with the structure of an \( \mathcal{O}_Y \)-algebra. Then \( X = \text{spec}_Y V \).

Question: Which vector bundles admit the structure of an \( \mathcal{O}_Y \)-algebra?

Suppose \( V = f_\ast \mathcal{O}_X \). We have

\[ \mathcal{O}_Y \rightarrow V \quad (\text{char } k \not= \) 0 \]

So \( V = \mathcal{O}_Y \oplus E \rightarrow 0 \)

Answer: Any such \( V \) admits an algebra structure.

Take \( E \otimes E \rightarrow V \) to be zero.

\[ X = \text{Thickening of } E \]  
\& \text{zero section.}

Modified Q: \( X, Y \) smooth, connected.
Then $E$ exhibits positivity.

- $H^0(Y, E^v) = 0$
- For $Y = \mathbb{P}^n$, then $E$ is ample (Lazarsfeld)
- $E$ is weakly positive, so nef if $\dim Y = 1$.

(Peternell-Sommese)

Not sufficient.

Example: $Y = \mathbb{P}^1$, $f: X \to Y$

Then $E = \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_{d-1})$,

where $a_1, a_2, \ldots, a_{d-1} > 0$.

Called "scrollar invariants" of $X$.

$d=2$: Any $a_1 > 0$ can be a scrollar invariant.

$d=3$: $a_1 \geq a_2 > 0$ are scrollar invariants iff

$2a_2 \geq a_1 \geq a_2$.

In general, a necessary condition for $a_1, \ldots, a_{d-1}$ to be scrollar invariants is that they are not "too far apart." (Ohbuchi, Coppins, Mortens).

E.g. (Ohbuchi) $\Rightarrow$ (d-1) $a_{d-1} \geq a_1$ (having some exceptions)

Asymptotic $Q$: Does every $E$ arise from a finite cover up to twisting by a line bundle?

E.g. $\mathcal{O}(1) \oplus \mathcal{O}(99) \rightarrow \text{NO}$

$\mathcal{O}(1001) \oplus \mathcal{O}(1099) \rightarrow \text{YES!}$
Thm 1: Let $Y$ be a smooth curve and $E$ a v.b on $Y$. There exists $N$ (depending on $Y, E$) such that for every line bundle $L$ of degree $\geq N$, the twist $E \otimes L$ arises from a cover $f: X \to Y$ with smooth $X$.

Let $\mathcal{H}_{d,g}(Y) = \left\{ f: X \to Y \mid g(x) = g, \deg f = d \right\}$

$f \mapsto E_f$ of $\text{rk}(d-1) \& \deg b = g-1 - d(g-1)$.

$\mathcal{M}_{d-1, b} = \left\{ \text{v.b. } b \mid \text{rk } d-1 \& \deg b \text{ on } Y \right\}$

Thm 2: Suppose $g(Y) \geq 2$. If $g$ is sufficiently large, then a general $f \in \mathcal{H}_{d,g}(Y)$ gives a stable $E_f$.

Also, the map

$$\mathcal{H}_{d,g}(Y) \longrightarrow \mathcal{M}_{d-1, b}(Y)$$

$$f \longmapsto E_f$$

is dominant.

Rem: Thm 2 proved by Kaner for $d \leq 5$ (2004,05,13). Using explicit structure theorems for coverings of deg $\leq 5$.

No such theorems for $d \geq 6$!

Q: Effective bounds ?
Ideas behind the proof
**Thm.** Every projective bundle $E$ arises from $X \rightarrow Y$ with smooth $X$.

**Step 1:**

$$E = L_1 \oplus \ldots \oplus L_{d-1}, \quad \deg L_i \gg \deg L_{i+1}$$

$$X \xrightarrow{\{i;i;i\}} Y \quad \{d \text{ copies}\}$$

**S**

$$X = \{x_i\} \quad \{X_{x_i}\}$$

$$Y. \downarrow \downarrow f$$

$$0 \rightarrow L_i \rightarrow f_* O_{X_{x_i}} \rightarrow f_* O_{X_i} \rightarrow 0$$

$$0 \rightarrow L_i \rightarrow ? \rightarrow \mathcal{O} \oplus L_1 \oplus \ldots \oplus L_{i-1} \rightarrow 0$$

**Must be split because** $\deg L_i \ll \deg L_j$ for $j < i$.

**So inductively**

$$f_* O_X = O_Y \oplus L_1 \oplus \ldots \oplus L_{d-1}$$

$$\Rightarrow E_f = L_1 \oplus \ldots \oplus L_{d-1}.$$ 

**But** $X$ is not smooth!
Basic fact:

\[ f : X \rightarrow Y \]

A finite flat map of degree \( d \)

Then we have a canonical embedding \( i \)

\[ X \rightarrow \mathbb{P}E_f \]

So

\[ \mathbb{P}E \]

Smooth out \( X \) inside \( \mathbb{P}E_f =: \mathbb{P} \)

But \( N_{X/\mathbb{P}} \) is typically negative.

Solution: \( X' = X \cup \{ \text{ rational normal curves} \} \).
Prop: For generic choices of Rational normal curves, $N_X/\mathbb{P}$ becomes sufficiently positive, i.e.

1) $H^0(N_X/\mathbb{P}) \rightarrow H^0(K_X)$ for some $X'$

2) $H^1(N_X/\mathbb{P}) = 0$.

Corollary: 1) $X$ is a limit of smooth $X_t \subset \text{PE}$ ($X_t$ give the same scroll $\text{PE}$).

2) $\text{brCov} \rightarrow \text{brBun}$ is smooth at $[X_t \rightarrow \mathbb{P}]$. So any scroll that is trivially deg. to $\text{PE}$ arises from br. covers.

Step 2: $\text{Hilb}/\Delta$ is smooth at $[X_t]$. So $X_t$ can be deformed into the gen. fiber.

Step 3: Any v.b. isotr. specializes to $L_1 \oplus \ldots \oplus L_{k-1}$.

$$\deg L_i \Rightarrow \deg L_{k-1}$$ (Exercise).
Higher dimensions.

Let $Y$ be a smooth projective variety $Y$ and $L$ an ample line bundle on $Y$.

Given a vector bundle $E$ on $Y$, $E^\otimes L^n$ arises from a finite cover for sufficiently large $n$.

Set $d = rE + 1$.

It is false if $\dim Y = d$.

Consider the multiplication 

$$E^\otimes E^\vee \to O \oplus E^\vee$$

Must be 0 for some $y \in Y$.

⇒ Fiber of $X \to Y$ over $y$ is a fat point.

Contradict $X$ is smooth (even Gorenstein).

Lazarsfeld ⇒ it is false for $Y = \mathbb{P}^r$, $r \geq d + 1$.

For $\dim Y \geq d + 1$, there are nontrivial restrictions on the topology of $X$.

Q: Is it true for $\dim Y \leq d$?