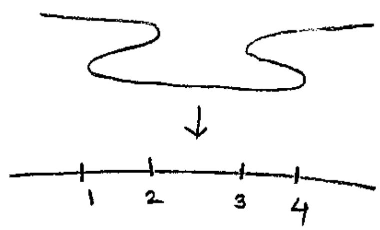
Alternate Compactifications of Hurwitz Spaces

$\mathbb{C}$ /

$$H^d_g = \left\{ \begin{array}{l} \varphi : C \xrightarrow{d:1} \mathbb{P}^1,\ p_1, \dots, p_b \\ C \text{ sm, genus } g \\ \varphi \text{ simply branched over } p_i \end{array} \right\}$$

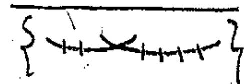
$b = 2g+2d-2$

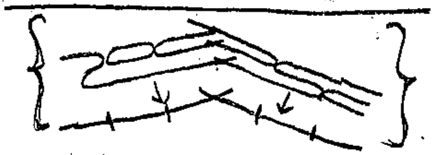


$$\begin{tikzcd} \overline{H}^d_g \arrow[rrr] \arrow[ddr] & & & \overline{M}_g \arrow[r] & \overline{M}_g^{cusps} \arrow[r, dashed, leftrightarrow] & \overline{M}_g^{tn} \cdots ? \\ & H^d_g \arrow[u, phantom, "\cup"] \arrow[rr] \arrow[d, "fin"] & & M_g \arrow[u, phantom, "\subset"] & & \\ & \overline{M}_{0,b} \supset M_{0,b} & & & & \end{tikzcd}$$

Compactifications

$\overline{M}_g = M_g \cup \overline{\{$  $\}} \cup \overline{\{ \ i \quad g-i \ \}}$

$\overline{M}_{0,b} = M_{0,b} \cup \overline{\{$  $\}}$

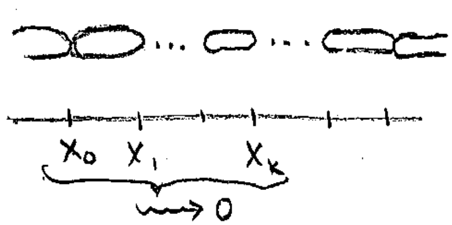
$\overline{H}^d_g = H^d_g \cup \overline{\{$  $\}} \cup \dots$

Goal: Explore other compact^n by allowing b.p to collide.

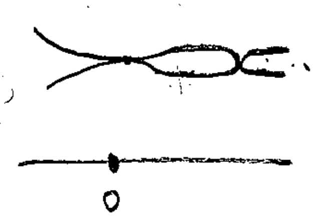
Aside:

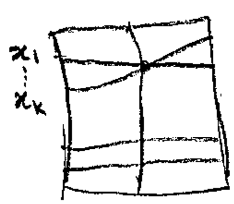
When branch points collide:

$y^2 = (x-x_1)\dots(x-x_k)\dots(x-x_n) \rightsquigarrow y^2 = x^k\cdot(\dots)$

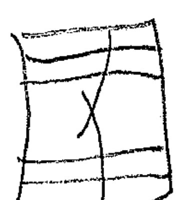


$\rightsquigarrow$



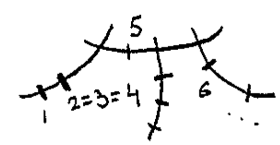


replace by



Weighted Pointed rational curves (Hassett)

$\omega = (\omega_1, \dots, \omega_b)$, $\omega_i \in \mathbb{Q}$, $0 < \omega_i \leq 1$, $\Sigma \omega_i > 2$

$\overline{M}_{0,b}(\omega) = \Bigg\{$  $\Bigg|$ Total wt of coincident pts ≤ 1; $\omega_c(\omega_1 p_1 + \dots \omega_b p_b)$ ample $\Bigg\}$

$\overline{M}_{0,b} \longrightarrow \overline{M}_{0,b}(\omega)$

$\overline{M}_{0,b} \ni$  $\rightsquigarrow$  $\in \overline{M}_{0,b}(\omega)$

Thm (Weighted adm covers).

The space

$$\overline{\mathcal{H}}_g^d(\omega) = \left\{ \begin{array}{l} \varphi : C \xrightarrow{d:1} P,\ p_1, \ldots, p_b, \\ (P, p_1, \ldots, p_b) \in \overline{\mathcal{M}}_{0,b}(\omega) \\ C \text{ conn, genus } g, \\ br(\varphi) = p_1 + \cdots + p_b \end{array} \right\}$$

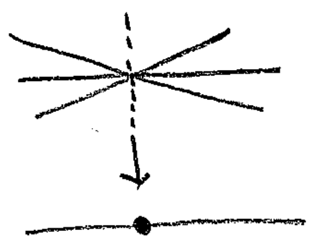
is a proper DM stack with a proj coarse sp. containing $\mathcal{H}_g^d$ as open subsp. & admitting

$$br : \overline{\mathcal{H}}_g^d(\omega) \to \overline{\mathcal{M}}_{0,b}(\omega).$$

$\square$

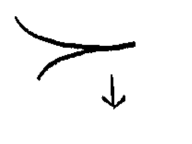
If ≥ 6 b.p can collide

$\rightsquigarrow$ br is not finite



planar trip pt

$d=2$: k br pt collide.



A_{k-1} sing.

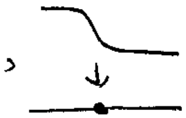
$y^2 = x^k$.

Examples:

2 br pts collide:

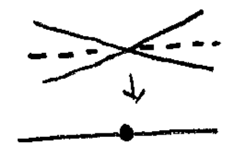
or

,



3 b.p. collide

4. b.p.



sp. trip pt.

Thm: For $d = 2, 3$, any ω

$\overline{\mathcal{H}}_g^d(\omega)$ is sm & irred.

Pf (Sketch):

$d=2 \rightsquigarrow$ planar sing ✓

$d=3 \rightsquigarrow$ spatial sing ✓.

$\square$

$d=2 \to$ Maksym Fedorchuk

"Spaces of hypell curves ..."

Spaces of Trig Curves (d=3).

$$\begin{array}{ccc} & & \mathbb{F}_M \\ & \nearrow & \downarrow \\ \text{trig genus } g \quad C & \xhookrightarrow{K_C} & \mathbb{P}^{g-1} \end{array}$$

Maroni invariant.

$\varphi: C \to \mathbb{P}^1$, $\varphi_* \mathcal{O}_C = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-a) \oplus \mathcal{O}_{\mathbb{P}^1}(-b)$

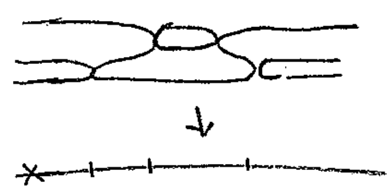
$a+b = g+2$, $a, b > 0$,

$M = |a-b|$, Upper semi cont, $\equiv g \pmod 2$

$T_g = H_g^3$, unordered br. pts.

$\omega = (1, \dots, 1), \dots, (\frac{2}{b}+\varepsilon, \dots, \frac{2}{b}+\varepsilon)$

Add one more marked pt.



$T_{g;1} = \left\{ \varphi: C \xrightarrow{3:1} \mathbb{P}^1, \sigma \;\; \sigma \notin br(\varphi) \right\}$.

$\omega_\varepsilon = (1; \varepsilon, \dots, \varepsilon)$, $1 \geq \varepsilon \geq \frac{1}{b-1}$

$\overline{\overline{T}}_{g;1}(\varepsilon) := \overline{\overline{T}}_{g;1}(\omega_\varepsilon)$.

$$\underset{\text{Adm covers}}{\overset{\overline{\overline{T}}_{g;1}(1)}{\parallel}} \dashrightarrow \underbrace{\overline{\overline{T}}_{g;1}(\tfrac{1}{2}) \dashrightarrow \cdots \dashrightarrow \overline{\overline{T}}_{g;1}(\tfrac{1}{b-1})}_{\substack{\text{Div contr.} \\ \text{Seq. contract bdary div.}}}$$

$\overline{\overline{T}}_{g;1}(\frac{1}{b-1}) = \{ \varphi: C \to \mathbb{P}^1, \sigma, \; \sigma \notin br(\varphi) \}$

Pic rk 3, not Fano

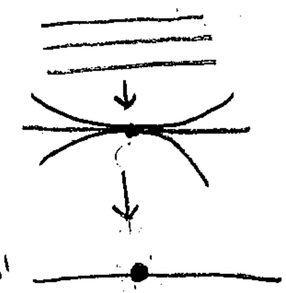
Punctually Ramified covers

$\varphi: C \xrightarrow{3:1} \mathbb{P}^1$, $br(\varphi) = b \cdot p$.

$C^\vee$

C

$\mathbb{P}^1$



$0 \to \varphi_* \mathcal{O}_C / \mathcal{O}_{\mathbb{P}^1} \to \varphi_* \mathcal{O}_{C^\vee} / \mathcal{O}_{\mathbb{P}^1} \to \underset{\text{supp at } p}{Q} \to 0$

$\text{length}(Q) = g+2$

$Q \cong k[t]/t^a \oplus k[t]/t^b$, $a, b > 0$, $a+b = g+2$

$\mathcal{M} := |a-b|$, $\equiv g \pmod 2$ lower semi cont.

Thm: Let $0 \le l \le g$, $l \equiv g \pmod 2$.

The space

$\overline{\mathcal{T}}^{\,l}_{g;1}$ of $\varphi: C \to \mathbb{P}^1, \sigma$ where

- C is a curve of genus g
- φ deg 3, $\sigma \notin br(\varphi)$
- Maroni$(\varphi) \le l$ and

  if $br(\varphi) = b \cdot p$ then $\mu(\varphi) > l$

is a sm, proper DM stack with a proj coarse space birat to $T_{g;1}$.

$\square$

Rmk: $\overline{T}^{\,g}_{g;1} = \overline{T}_{g;1}\left(\frac{1}{b-1}\right)$

$$\underbrace{\overline{T}_{g;1}(1) \dashrightarrow \cdots \dashrightarrow \overline{T}_{g;1}\left(\tfrac{1}{b-1}\right)}_{\text{Div contr}}$$

$\overline{T}_{g;1}(1)$ = Adm cov

$$\overline{T}_{g;1}\left(\tfrac{1}{b-1}\right) = \underbrace{\overline{T}^{\,g}_{g;1} \dashrightarrow \overline{T}^{\,g-2}_{g;1} \dashrightarrow \cdots \dashrightarrow \overline{T}^{\,0\text{ or }1}_{g;1}}_{✱}$$

Thm: ① $\overline{T}^{\,g}_{g;1} \dashrightarrow \overline{T}^{\,g-2}_{g;1}$ extends to a morphism that contracts the hyp.ell divisor H to a point corresp. to a D_{2g-2} sing.

$H \longmapsto$ [ $(y^2 - x^{2g})(y-x)$]

② For g even, $\overline{T}^{\,2}_{g;1} \dashrightarrow \overline{T}^{\,0}_{g;1}$ extends to a morphism and contracts the divisor of covers with Maroni $= 2$ to a $\mathbb{P}^1$.

③ All the other maps in ✱ are iso in codim 1.

The Last spaces

g even

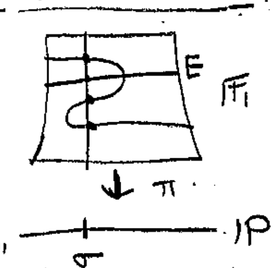
$\overline{T}^{\,0}_{g;1} \cong$ Wtd proj space $/ S_3$

Pic rk 2.

g odd

$\overline{T}^{\,1}_{g;1}$ Pic rk 2, fibered over $\mathbb{P}^1$

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d] & \mathbb{F}_1 \arrow[dl, "\pi"] \\ \mathbb{P}^1, \sigma & \end{tikzcd}$$



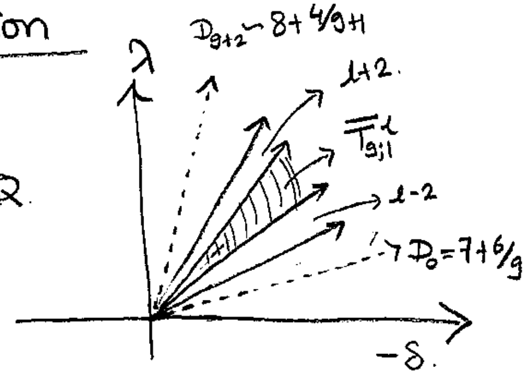
$[C \xrightarrow{\varphi} \mathbb{P}^1, \sigma] \longmapsto$ "Cross ratio" of 4 marked pts on $\pi^{-1}(\sigma)$.

Chamber decomposition

Take $0 < l < g$

$\mathrm{Pic}_{\mathbb{Q}} = \mathrm{Pic}(\overline{T}^{\,l}_{g;1}) \otimes \mathbb{Q}$

$= \langle \lambda, \delta \rangle$



Thm: The ample cone of $\overline{T}^{\,l}_{g;1}$ is bounded by D_l, and D_{l+2}, where

$$D_l = \{(7g+6)\lambda - g\delta\} + \frac{l^2}{g+2}(g\lambda - \delta).$$

- More pic
- H^1_2, H^2_2 $= M_{1,1}$, M_2 etc. in the intro
- Mention ~~#~~ the last Fano / fibration models.
- Draw $\overline{M}_g^{\,cusp}$, $\overline{M}_g^{\,tacnodes}$ on the board.
- Don't write out def. of adm covers. Just draw pictures.
- $y^2 = (x-x_0)\cdots(x-x_l)$ to describe br. pts coming together.

  & draw pic of adm covers.