Towards a birational Classification of algebraic varieties - Work of Caucher Birkar.

1. Algebraic Varieties.

Algebraic variety = Set of solutions of polynomial equations.

Ex. \( X = \{ (x, y, z) \mid x^2 + y^2 = z^2 \} \) \( \xrightarrow{\psi} \) \( \dim X \)

Solutions in \( \mathbb{Z} \) = \{ Pythagorean triples \}.

Solutions in \( \mathbb{R} \) = cone over \( S^1 \).

Solutions in \( \mathbb{C} \) = complex cone over \( S^2 \).

Isomorphisms

An isomorphism between two complex varieties is a bijection \( \psi : Y \rightarrow X \)

\( \psi \) and \( \psi^{-1} \) are defined by polynomials.

Ex. \( Y = \{ (x, y, z) \mid xy = z^2 \} \) \( \xrightarrow{\sim} \) \( X \)

Via \( \psi : x \mapsto x + iy \), \( y \mapsto x - iy \), \( z \mapsto z \).
Problem: Describe all isomorphism classes.

2. Birational algebraic geometry.

A birational iso $\varphi: Y \sim \rightarrow X$ is a bijection $\varphi: U \rightarrow V$ between a dense open $U \subset X$ & $V \subset Y$ such that $\varphi$ & $\varphi^{-1}$ are defined by rational functions.

Ex: $\mathbb{C} \sim \rightarrow Y$ by

$$(s, t) \mapsto (s, \frac{t^2}{s}, t)$$

$$(x, z) \leftarrow (x, y, z)$$

Problem: Describe all birational iso classes.

1. Identify a distinguished element in each birat iso class ("canonical model")

2. Describe the canonical models.

$\dim 1$

1. There is a unique smooth & compact $(\mathbb{P}^1)$. $X$ in every birat class.

2. $g=0$ $g=1$ $g=2, 3, 4, \ldots.$

$\mathbb{P}^1$ 1 dim family $3g-3$ dim family.
3. The minimal model program

2. The canonical class.

\( X \) an alg. variety.

3. Distinguished element \( K_X \in H^2(X, \mathbb{Q}) \).

\[ K_X = c_1(\Omega_X) \quad \Omega_X = \text{Holomorphic cotangent bundle.} \]

Ex.

\[ \dim X = 1, \ X \text{ smooth compact, } H^2(X) \cong \mathbb{Q}. \]

\[ K_X = 2g-2. \]

- \( g=0 \) : \( K_X < 0 \)
- \( g=1 \) : \( K_X = 0 \)
- \( g=2 \) : \( K_X > 0 \)

4. The trichotomy.

\( X \) is

1. Fano if \( K_X \) is ant ample (\( K_X \cdot C < 0 \forall C \))
2. Calabi-Yau if \( K_X \) is trivial. (\( K_X \cdot C = 0 \forall C \))
3. Canonically polarised if \( K_X \) is ample (\( K_X \cdot C > 0 \forall C \)).

- spherical: small/trivial \( \pi_1 \), many \( \mathbb{Q} \) pts., big aut.
- flat: close to abelian \( \pi_1 \), many but not too many.
- hyperbolic: complicated \( \pi_1 \), few \( \mathbb{Q} \) -points, finite aut.
MMP - Up to birat. iso. every $X$ can be broken down into these 3 archetypes.

$X \rightarrow X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow \ldots \rightarrow X_n$

Each step @ div. contraction or flip.

$X_n =$ can. polarized or

$X_n$ with Fano/CY fibers.

$\downarrow$ lower dim

$\dim X = 1$ - $X = X_n$.

$\dim X = 2$ - Only div. contr. needed. All $X_i$ are smooth.

(Castelnuovo-Enriques, Early 1900)

$X \rightarrow X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow \ldots$

Contract $C$ such that $K \cdot C < 0$.

In $\dim \geq 3$, introduces singularities.

Identify a class of singular $X$ - a factorial terminal.

Preserved under divisorial contraction, but not small.

Flip = a surgery that improves singularities does not introduce $K$-neg curves.

$\dim = 3$ : Flips exist.

There cannot be an inf. seq. of flips.

$\Rightarrow$ MMP terminated + $X_n$ is as expected.
Higher dim: Birkar, Cascini, Hacon, McKernan (2012)

1. Flips exist.
2. MMP terminates, if \( X \) is of general type (if flips are carefully chosen).
   (In this case \( X_n \) is canonically polarised.)

Conj: 1. MMP terminates in general.
       (Termination of flips).
2. If \( X_n \) is not canonically polarized, then it admits a CY fibration (Abundance).

Boundedness results - BAB conjecture.

Thm (Birkar). The class of Fano \( X \) of given dim with canonical \( A \)-terminal singularities form a finite dim family.

even \( \epsilon \)-log canonical for a given \( \epsilon > 0 \).

+ relative versions
+ existence of complements (nice elements in \( 1 - mK_X \) for bounded \( m \)).

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   Flip \( X \) \rightarrow \text{"Relative Fano"}
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Thm \( \Rightarrow \) control over flips.

Aside: Boundedness of canonically polarized \( X \) of a given dim & volume is also known
       (Tsuji, Hacon-McKernan, Takayama).