Classical Hodge Theory – Anand Feb 16, 2017

Ref: Sabbah–Schnell, MHM project

X smooth, projective var$/\mathbb{C}$ & $H^\bullet(X,\mathbb{Z}) = \bigoplus_k H^k(X,\mathbb{Z})$

Thm: $H^k(X,\mathbb{C}) \simeq \bigoplus H^{p,q}(X)$, where $H^{p,q}(X) = H^q(X, \Omega_X^p)$
$\Omega_X^p = \Lambda^p$ (Holomorphic cotangent bundle)
$= \{$ classes represented by closed p,q forms $\}$
locally, $f\, dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$

$H^{p,q} = \overline{H^{q,p}}$

More algebraic way:

$$\underline{\mathbb{C}} \longrightarrow \mathcal{O}_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \Omega_X^2 \longrightarrow \cdots \longrightarrow \Omega_X^n \longrightarrow 0$$

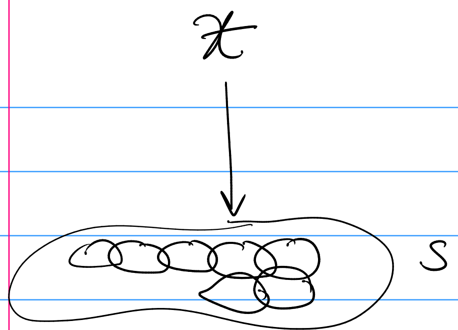
(hol./alg.)

This sequence is exact.
Ω_X^i not flabby / cohomologically trivial if we take holomorphic differentials.
$\Rightarrow$ Get a spectral seq $H^q(\Omega_X^p) \Rightarrow H^k(X, \underline{\mathbb{C}})$ } Hodge–de Rham spectral seq.
Thm: This seq is degenerate (all boundary maps are zero)

Hodge structures

Def 0: A $(\mathbb{Z}/\mathbb{Q}/\mathbb{R})$-Hodge structure $=: A$ of wt k consists of a free A-module H, along with a decomposition
$H \otimes_A \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}$, such that $H^{p,q} = \overline{H^{q,p}}$.

Families: Let $\pi: \mathcal{X} \to S$ be a family of smooth complex projective varieties. [For now let $A = \mathbb{Q}$ to make notation easier.]
Set $H_{\mathbb{Q}} = R^k\pi_*(\underline{\mathbb{Q}})$, which is a local system on S.



$\mathbb{Q}$-vector bundle / S whose transition maps are constant (locally, H^0(preimage of open set))

Get $H_{\mathbb{C}} = H_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$: a flat complex vector bundle

flat means either:

(1) constant transition functions, or

(2) a notion of local constancy, or

(3) $\mathbb{C}$-vb with a flat connection ∇

(4) A rep of $\pi_1(S)$

To get the connection, choose a flat basis of H: $e_1, \dots, e_n$

$\nabla(f \otimes e_1) = df \otimes e_1$, and $\nabla^2 f = 0$ b/c $d^2 = 0$

Conversely, given H with a flat connection, (ie $\nabla^2 = 0$) then local constancy means: s is locally constant if $\nabla(s) = 0$.

Hodge structure on $H_{\mathbb{C}}$:

$$H_{\mathbb{C}} \simeq \bigoplus_{p+q=k} \underbrace{R^q \pi_*(\Omega^p_X)}_{\text{not necessarily flat v.b.}} .$$

Rmk: $R^q \pi_*(\Omega^p_X) \subset H_{\mathbb{C}}$ is a sub-bundle. Both bundles have holomorphic structures, but it is NOT a holomorphic bundle. However it is a C^∞-sub-bundle.

$$H^{k,0} \oplus H^{k-1,1} \oplus \cdots \oplus H^{1,k-1} \oplus H^{0,k}$$

$\vdots$

$F_1^{\bullet}$, where $F_1^i = \bigoplus_{j \geq i} H^{j,k-j}$

$F_\bullet^i \subset H_{\mathbb{C}}$ is holomorphic (Theorem)

$$H^{k,0} \oplus H^{k-1,1} \oplus \cdots \oplus H^{1,k-1} \oplus H^{0,k}$$

$$F_2^i = \bigoplus_{j \geq i} H^{k-j,j}$$

$F_1^\bullet = \overline{F_2^\bullet}$ & $F_2^\bullet \subset H_{\mathbb{C}}$ is anti-holomorphic

S a variety. A variation of Hodge structures (VHS) of wt k on S consists of:

★ (i) A $\mathbb{Q}$-local system $H_{\mathbb{Q}}$ / (i')★ such that $H_{\mathbb{C}}$ flat vb.

(ii) Two filtrations $F_1^\bullet$ & $F_2^\bullet$ of $H_{\mathbb{C}}$, such that:

★ (a) $F_1^\bullet = \overline{F_2^\bullet}$

(b) k-opposedness property: $F_1^i \cap F_2^j = 0$ if $i+j \geq k$, and $F_1^i \oplus F_2^{k-i} \xrightarrow{\sim} H_{\mathbb{C}}$

[Rmk: Given $F_1^\bullet$ & $F_2^\bullet$, we can construct $H^{p,q}$ by taking $(F_1^p \cap F_2^q)$]

(c) $F_1^\bullet \subseteq H_{\mathbb{C}}$ is holomorphic, and $F_2^\bullet$ is anti-holomorphic

(d) [Griffiths transversality]: $F_1^p \otimes T_S \xrightarrow{\nabla} F_1^{p-1}$ & similar for $F_2^\bullet$ "anti-Griffiths-transversality": $F_2^p \otimes \overline{T}_S \xrightarrow{\nabla} F_2^{p-1}$

[Rmk: (d) is a theorem in the geometric context, but in the general setting we need this condition.]

[Rmk: $F_2^\bullet$ is automatic after imposing a $\mathbb{Q}$-structure & that $F_1^\bullet = \overline{F_2^\bullet}$, but not otherwise.]

Look ahead:

★ 1) $\mathbb{Q}/\mathbb{R}$ - structure ⤳ Perverse sheaves
★ 2) $\mathbb{C}$ - Hodge structure ⤳ D-modules

→ Riemann–Hilbert correspondence.

Operations:

(1) $H_1 \otimes H_2$

wt k_1, wt k_2 → total wt $k_1 + k_2$

(2) $\mathrm{Hom}(H_1, H_2) \to$ wt $k_2 - k_1$

(3) Dual : $\mathrm{Hom}(H, \mathbb{C}) \longrightarrow$ wt $-k$

trivial Hodge structure

(4) Conjugation : $(\overline{H}, \overline{\nabla}, \overline{F_2^\bullet}, \overline{F_1^\bullet})$ of wt k

(5) Adjoint := dual of conjugate → wt $-k$

(6) Shift : $(H, F_1^\bullet[m], F_2^\bullet[\ell]) \to$ wt $k-m-\ell$

$F_1^i[m] = F_1^{i+m}$

Usually, shift by (m,m) so that $F_1^\bullet = \overline{F_2^\bullet}$ remains true.

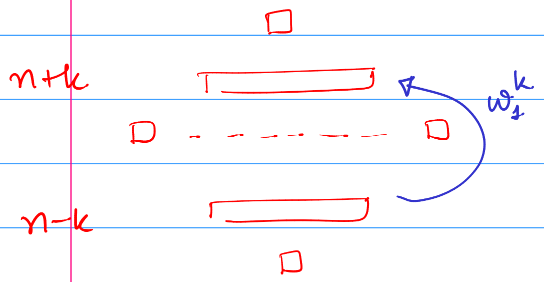
aka Tate twist by m. ⤳ wt $k-2m$.

Polarization:

Defn: A polarization on H is a nondegenerate, flat bilinear form $H \otimes \overline{H} \longrightarrow \underline{\mathbb{C}}(-k)$, a degree-0 map of HS.

$$X \text{ } n\text{-dimensional} \Rightarrow \underset{(\text{wt } n)}{H^n(X,\mathbb{C})} \otimes \underset{(\text{wt } n)}{H^n(X,\mathbb{C})} \longrightarrow \underbrace{\underline{\mathbb{C}}(-n)}_{\text{wt } 2n}$$

flat means: $e_1 \otimes e_2 \to$ [locally constant function]



L ample line bundle

$\omega = c_1(L,L)$ degree $(1,1)$

Hard Lefschetz : ω^k is an iso.