APPENDIX TO “CYCLIC COVERING MORPHISMS ON $\overline{M}_{0,n}$”
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APPENDIX A. Class of the Hodge eigenbundle using orbifold Riemann–Roch

In [2] Section 6], the class of the Hodge eigenbundle $E_j$ was computed indirectly by calculating it on the $F$-curves. We describe a more direct method using the Grothendieck–Riemann–Roch theorem for Deligne–Mumford stacks. Throughout, $\mu_r$ denotes the group of $r$th roots of unity and $C(i)$ the line bundle on $B\mu_r$ associated to the character $\zeta \mapsto \zeta^i$.

A.1. Expressing $E_j$ as a pushforward. Let $p \geq 2$ be an integer and $(d_1, \ldots, d_n)$ a sequence of non-negative integers such that $\sum d_j = mp$ for some integer $m$. Consider the cyclic cover $\phi: C \to P^1$ given by the regular projective model of

$$ y^p = (x - x_1)^{d_1} \cdots (x - x_n)^{d_n}. $$

In these coordinates, $O_C$ is generated as an $O_{P^1}$ module by the functions $1, y/f_1, \ldots, y^{p-1}/f_{p-1}$ where $f_j = (x - x_1)^{[j d_1/p]} \cdots (x - x_n)^{[j d_n/p]}$. Note that these generators are $\mu_p$ eigenvectors. Letting $D$ be the $Q$ divisor $D = \sum(d_i/p)x_i$, we thus have the $\mu_p$-equivariant decomposition

$$ \phi_*O_C = \bigoplus_{j=0}^{p-1} O_{P^1}(-jm) \otimes O_{P^1}([jD]), $$

where the local generator of the $j$th summand is $y^j/f_j$. We then get the $\mu_p$-eigenspace decomposition

$$ H^1(C, O_C) = \bigoplus_{j=0}^{p-1} H^1(P^1, O_{P^1}(-jm) \otimes O_{P^1}([jD])). $$

The above analysis for an individual cover goes through over the moduli stack. Let $M$ be the stack of $p$-divisible $n$-pointed orbicurves with the universal curve $\pi: P \to M$ with $n$ sections $\sigma_i$ and let $L$ be a line bundle on $P$ with $L^p = O_P(\sum d_i \sigma_i)$. On $M$ lives the universal cyclic $p$ cover $\phi: C \to P$ whose fibers are as above. Setting $D = \sum(d_i/p)\sigma_i$, we get the the $\mu_p$-equivariant decomposition

$$ \phi_*O_C = \bigoplus_{j=0}^{p-1} L^{-j} \otimes O_P([jD]). $$

Applying $R\pi_*$ gives the $\mu_p$-eigenbundle decomposition

$$ R^1\pi_*(O_C) = \bigoplus_{j=0}^{p-1} R^1\pi_*(L^{-j} \otimes O_P([jD])). $$

Since the left hand side is the dual of the Hodge bundle, we get

$$ E_j = R^1\pi_*(L^{-j} \otimes O_P([jD]))^*. $$

A.2. Applying Grothendieck–Riemann–Roch. Equation 1 reduces the computation of $c_1(E_j)$ to the familiar problem of computing the Chern classes of a (derived) push-forward. Since the push forward is from a stack, we have to use a bit of technology. For less cluttered notation, we calculate in the following setting: Let $B$ be a smooth projective curve, $\pi: P \to B$ a generically smooth family of rational orbinodal curves with $P$ smooth, and $F$ a line bundle on $P$. For an orbinode $x \in P$ with stabilizer $\mu_{r_x}$, let $F|_x \cong C(i_x)$, where $0 \leq i_x < r_x$. Denote by $\omega$ the class of the relative dualizing sheaf of $P \to B$. 

1
Proposition A.1. In the above setup, we have

\[ c_1(R\pi_* F) = \frac{c_1(F)^2}{2} - \frac{c_1(F) \cdot \omega}{2} - \sum_{\text{Orbi } x \in \mathcal{P}} \frac{i_x(r_x - i_x)}{2r_x} \]

Proof. By the GRR for Deligne–Mumford stacks [1, Corollary 5.3], we have

\[ \text{ch } R\pi_* F = \pi_* \left( \left( t \left( \frac{f^* F}{\lambda_1 N_f^*} \right) \right) \cdot \text{td}(IP) / \text{td}(B) \right), \]

where \( f : IP \to \mathcal{P} \) is the inertia, \( N_f^* \) is the conormal bundle of \( f \), the operator \( \lambda_1 \) of \( K(IP) \) is the alternating sum of wedge powers \( 1 - \Lambda^1 + \Lambda^2 + \ldots \) and the operator \( f \) on \( K(IP) \otimes C \) is the ‘twisting operator’ [1, § 4]. We evaluate these objects in our setup. First, note that the coarse space map \( \mathcal{P} \to P \) is an isomorphism except possibly at the singular points of \( P \to B \), where the local picture is

\[ \text{Spec } k[u,v]/(uv - t)/\mu_r \to \text{Spec } k[x,y]/(xy - t^r), \]

with \( \mu_r \) acting by \( u \to \zeta u \) and \( v \to \zeta^{-1} v \). Since the orbistructure is at isolated points, the inertia stack is

\[ IP = \mathcal{P} \sqcup \bigsqcup_{\text{Orbi } \chi \in \mathcal{P}} (\mu_r \setminus \{ \text{id} \}) \times B\mu_r. \]

The normal bundle \( N_f \) is trivial on the \( \mathcal{P} \) component and is \( C(1) \oplus C(-1) \) on the \( \{ \zeta \} \times B\mu_r \) component. The twisting operator acts trivially on the \( \mathcal{P} \) component and sends \( C(0) \) to \( \zeta^2 C(0) \) on the \( \{ \zeta \} \times B\mu_r \) component. Thus, [Equation 2] becomes

\[ \text{ch } R\pi_* F = \pi_* \left( 1 + c_1(F) + \frac{c_1(F)^2}{2} \right) \left( 1 - \frac{c_1(\Omega_{P/B})}{2} + \frac{c_1^2(\Omega_{P/B}) + c_2(\Omega_{P/B})}{12} \right) + \sum_{\text{Orbi } x \in \mathcal{P}} \frac{r_x(2 - \zeta - \zeta^{-1})}{r_x(2 - \zeta - \zeta^{-1})}. \]

Let \( \omega_{P/B} \) be the relative dualizing sheaf. Let \( \delta_{P/B} \) be the class of the support of \( \text{coker}(\Omega_{P/B} \to \omega_{P/B}) \) (with multiplicities) so that \( c_2(\Omega_{P/B}) = \delta_{P/B} \). Define \( \omega_{P/B} \) and \( \delta_{P/B} \) analogously. Identifying the rational Chow groups of \( \mathcal{P} \) and \( P \), we have \( c_1(\omega_{P/B}) = c_1(\omega_{P/B}) = \omega \). The value of \( \delta \), however, is different for \( \mathcal{P} \to B \) and \( P \to B \): the numerical contribution of the orbinode in [Equation 3] is \( 1/r \) whereas that of the node is \( r \). With these simplifications, we get

\[ c_1(R\pi_* F) = \frac{c_1(F)^2}{2} - \frac{c_1(F) \cdot \omega}{p} + \frac{\omega^2 + \delta_{P/B}}{12} + \sum_{\text{Orbi } x} \frac{1}{r_x - r_x} + \left( \sum_{1 \neq \zeta \in \mu_r} \frac{\zeta^i}{r_x(2 - \zeta - \zeta^{-1})} \right). \]

Since \( P \to B \) is a family of rational curves, we have \( \omega^2 + \delta_{P/B} = 0 \). The following is a nice exercise

\[ \sum_{1 \neq \zeta \in \mu_r} \frac{\zeta^i}{2 - \zeta - \zeta^{-1}} = \frac{i(i - r)}{2} + \frac{r^2 - 1}{12} \quad \text{(for } 0 \leq i \leq r). \]

Substituting in [Equation 4] gives the result. \( \square \)

Proposition A.2. The class of the Hodge eigenbundle is given by

\[ c_1(E_i) = \frac{1}{2p^2} \left( \sum_i \langle jd_i \rangle_p (p - jd_i)_p \psi_i - \sum_{I,j} \langle jd(I) \rangle_p \langle jd(J) \rangle_p \Delta_{I,J} \right). \]

Proof. Take a smooth curve \( B \) and a map \( B \to \mathcal{M} \) transverse to the boundary. Let \( \pi : \mathcal{P} \to B \) and \( \mathcal{L} \) be pullbacks from \( \mathcal{M} \). Set \( F = \mathcal{L}^{-1} \otimes \mathcal{O}_P(\langle j \rangle D) \). Then \( c_1(F) = - \sum_{i} (\langle jd_i \rangle_p/p) \cdot \sigma_i \). Let \( x \in \mathcal{P} \) be an orbinode corresponding to the boundary divisor \( \Delta_{I,J} \). Set \( d = \gcd(p, jd(I)) \). Then the stabilizer at \( x \) has order \( r = p/d \) and \( F|_x \cong C(i) \) for \( i = \langle jd(I)/d \rangle_r \). Also, \( x \) contributes \( r \) towards \( \Delta_{I,J} \). Using [Equation 1] and
Proposition A.1 we conclude that

\[ c_1(E_j) = c_1(R\pi_* F) = \frac{1}{2p^2} \left( \sum_i \langle jd_i \rangle_p^2 \sigma_i^2 + p \langle jd_i \rangle_p \sigma_i \cdot \omega - \sum_{\text{Orbi}_x} \frac{p^2 \langle jd(I)/d \rangle_r \langle jd(J)/d \rangle_r}{r} \right) \]

\[ = \frac{1}{2p^2} \left( \sum_i \langle jd_i \rangle_p \langle p - jd_i \rangle_p \psi_i - \sum_{I,J} \langle jd(I) \rangle_p \langle jd(J) \rangle_p \Delta_{I,J} \right). \]

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References