The Thurston compactification for a generic non-algebraic K3 surface

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Let X be a generic non-algebraic K3 surface, so that $\mathrm{Pic} X = 0$. Then the Mukai lattice of X is the hyperbolic lattice $\mathbf{Z} \oplus \mathbf{Z}$. The Mukai vector of a sheaf E is given by

$$v(E) = (\operatorname{rk} E, \operatorname{rk} E - c_2(E)).$$

The only spherical object, up to shifts, is the structure sheaf O_X. Let T denote the spherical twist in O_X.

Let k_x denote the skyscraper sheaf at a point $x \in X$. Changing x has no implications on anything, so it will be harmless to drop it from the notation. It will also be convenient to set

$$I_0 = k, \text{ the skyscraper sheaf at a point, and}$$
$$I_n = T^n I_0.$$

In particular, we have $I_1 = I[1]$, where I is the ideal sheaf of a point. Note that, in the Grothendieck group, we have

$$[I_n] = [I_{n+2}]$$

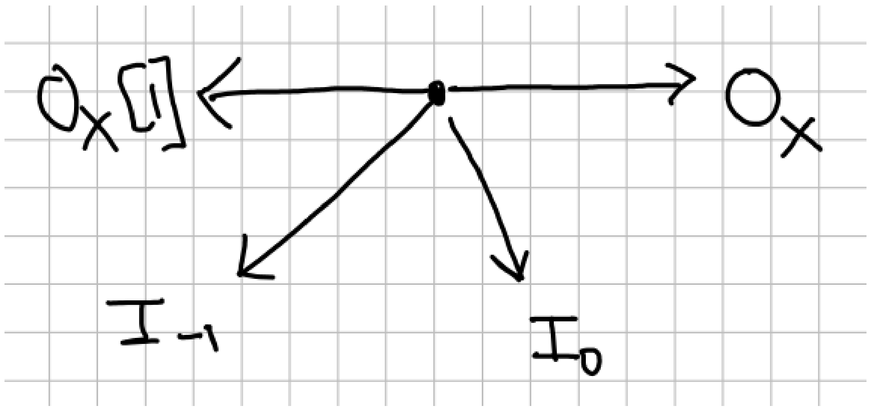
and

$$[I_0] - [I_1] = [O_X].$$

Our goal is to describe $\mathbf{P} \operatorname{Stab} X$ and its Thurston compactification. Ordinarily, we take the Thurston compactification to be the closure in the infinite projective space whose coordinates are indexed by the spherical objects. Since there is only one spherical object, we must enlarge the set of objects. A convenient choice is the set

$$S = \{O_X, I_n \text{ for } n \in \mathbf{Z}\}.$$

We summarize the description of the stability conditions from [1]. Every stability condition has the form $T^n\sigma$, where σ is a standard stability condition. For us, the standard stability conditions correspond to the ones with $z \in R^+ \cup R_0$ in the notation of [1, Section 4]. Let σ be a strictly standard stability condition, namely, one corresponding to $z \in R^+$. Here is a pictorial representation of σ:



The HN-factors of I_n for $n \geq 1$, listed in decreasing order of phase, are

$$I_n \sim I_0 + O_X[1] + \cdots + O_X[2-n],$$

and those for I_{-n} for $n \geq 2$ are

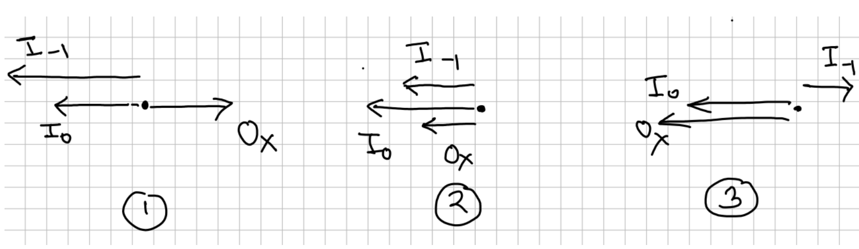
$$I_{-n} \sim O_X[n] + \cdots + O_x[2] + I_{-1}.$$

Let $m(O_X) = c$, $m(I_0) = b$, and $m(I_{-1}) = a$. Then the σ-mass vector for the objects $O_X, \ldots, I_{-3}, I_{-2}, I_{-1}, I_0, I_1, I_2, \ldots$ is

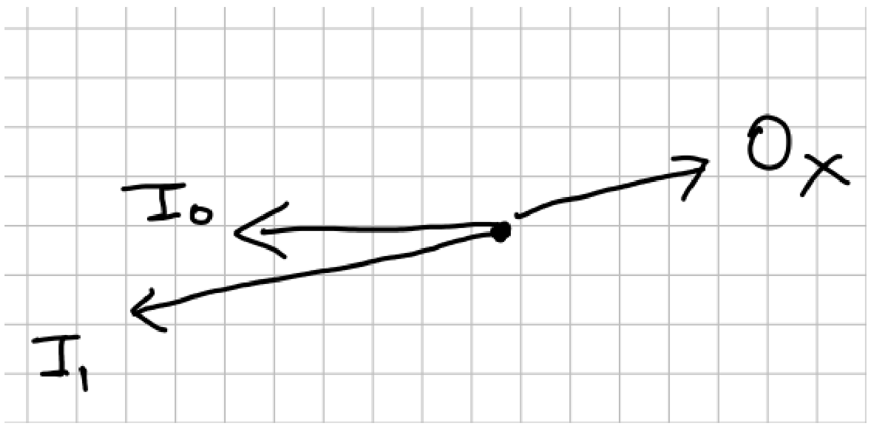
$$m(\sigma) = [c : \cdots : a + 2c : a + c : a : b : b + c : b + 2c : \ldots].$$

The a, b, c satisfy a (strict) triangle inequalities. The mass functional embeds this (open) triangle into the infinite projective space.

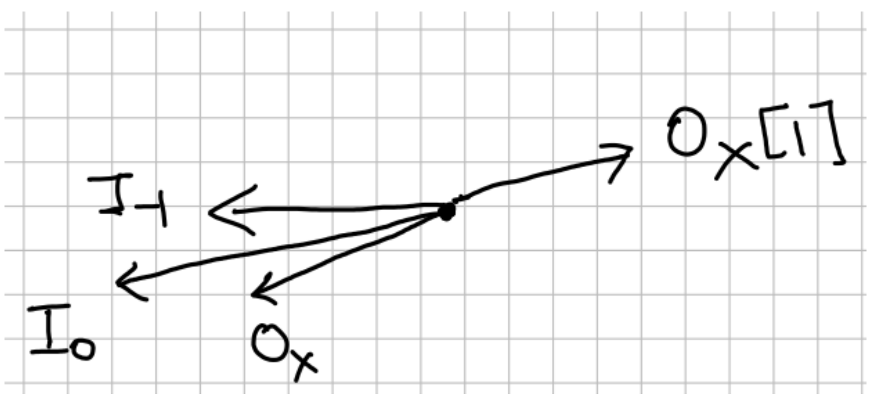
The three edges of the triangle correspond to degenerations of the triangle inequality. In terms of the central charge, they correspond to the following three types of pictures:



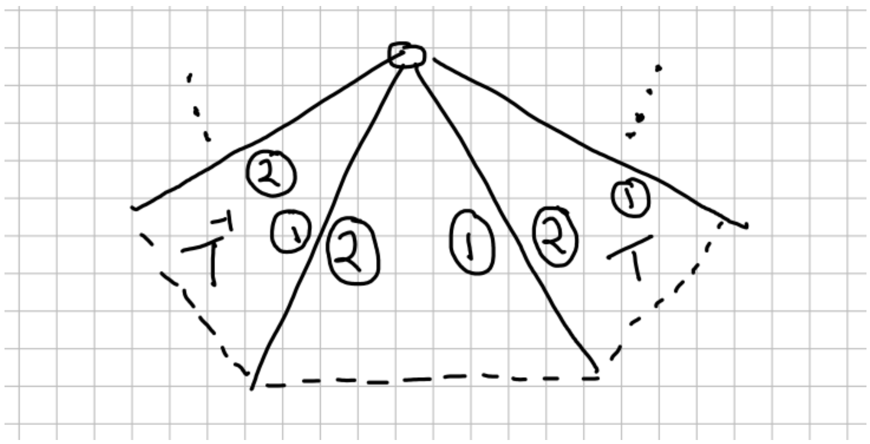
. Of these, the type (1) wall corresponds to $z \in R_0$ and forms the boundary between standard stability conditions and the T-translates of standard stability conditions. A stability condition on the other side of this wall looks like this:



. This is the T-translate of the following standard stability condition:



, which corresponds to a condition near a type (2) wall. We thus have the following picture:



. Topologically, this is isomorphic to $(0, 1) \times \mathbf{R}$.

Let us now think about the mass functionals. Approaching the vertex $(1) \cap (2)$ is equivalent to taking the mass c of O_X to 0. Then, by the triangle inequality, the masses a and b have to be equal, and hence, projectively, we get the point $[0 : \dots 1 : 1 : 1 : \dots]$. Note that this point is T-invariant, and hence is a common vertex of all the triangles. It corresponds to the functional $\overline{\hom}(O_X, -)$.

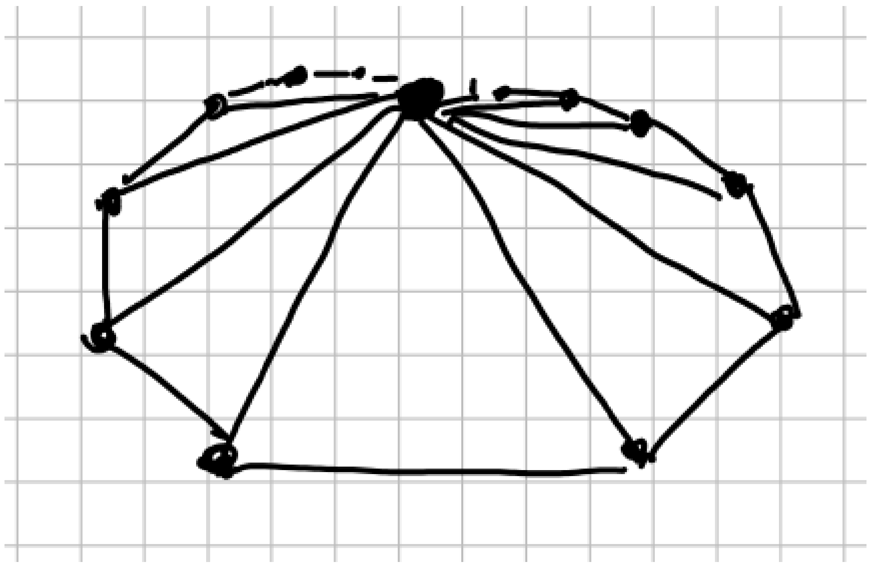
Approaching the vertex $(1) \cap (3)$ is equivalent to taking the mass a of I_0 to 0. Then the limiting point in the projective space is $p_0 = [1 : \dots : 2 : 1 : 0 : 1 : 2 : 3 : \dots]$. where the 0 is in the 0-th place.

How can we interpret the functional corresponding to p_0? I do not think it is a $\overline{\hom}$ functional, but it is an "occurence" functional. It "counts" the occurences of O_X in the (minimised) complex that describes the object. In other words, it is the rank of the complex at the generic point. (By the rank of a complex, we mean the sum of the ranks of its cohomology sheaves.)

The point p_0 is not T-stable. Let $p_n = T^n p_0$. Then the edge p_{-1}–p_0 is the edge of type (3) in the standard triangle. Its points correspond to the mass c of O_X equalling $a+b$, and the corresponding point in projective space is

$$(a + b : \dots : 2b + 3a : b + 2a : a : b : 2b + 3a : 3b + 4a : \dots) = ap_{-1} + bp_0.$$

Thus, the compactified $\mathbf{P}\,\mathrm{Stab}$ looks like this:

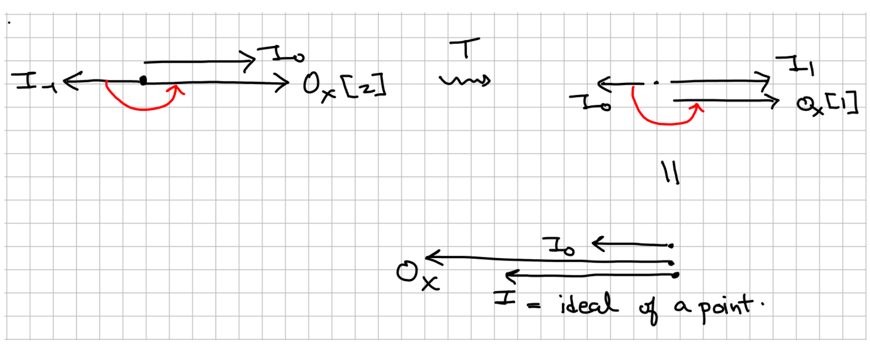


So, topologically, the compactification is a disk, whose interior is $\mathbf{P}\,\mathrm{Stab}$. The boundary is a circle with a distinguished point which is $\overline{\mathrm{homBar}}(O_X, -)$.

Why is it impossible to cross a type (3) wall

The picture indicates that it is not possible to cross the type (3) wall in the stability manifold. Let us try to understand why this should be the case.

Consider what happens on the type (3) wall. It is easier to think after applying T (and writing the objects with correct shifts!)



The resulting stability condition violates the local finiteness condition. How? It is easier to see that it violates the "support property", namely, there are semistable objects whose classes approach 0. Let m and n be positive integers with $n > m$. Let Q be the structure sheaf of n generic points and let $F_{m,n}$ be the kernel of a generic map

$$O_X^m \to Q.$$

Then $F_{m,n}$ is (probably) semi-stable, and of the same phase as O_X. But for appropriate choices of m and n, its class approaches 0.

Thus, the non local finiteness follows using the support property for Bridgeland stability conditions, but it would be nice to get an explicit infinite chain as well.

References

[1] Daniel Huybrechts, Emanuele Macrì, and Paolo Stellari. Stability conditions for generic $K3$ categories. *Compos. Math.*, 144(1):134–162, 2008.