THE SYZYGY STABILITY OF CURVES OF GENUS 7

ANAND DEOPURKAR

We show the following about the syzygy points of genus 7 curves. (See [DFS14] for the definition of syzygy points.)

1. A general curve of genus 7 has a stable first syzygy point.
2. A general tetragonal curve of genus 7 has (at least) a semistable first syzygy point.
3. A Casnati–Ekedahl special tetragonal curve of genus 7 has a strictly semistable first syzygy point. The syzygy points of all such curves coincide and this syzygy point is the syzygy point of the anti-canonically embedded del Pezzo surface of degree 6.

1. Generic stability

Proposition 1. A general curve of genus 7 has a stable first syzygy point.

Proof. It suffices to exhibit one canonical curve with stable first syzygy point. Let $C$ be the Fricke–Macbeath curve of genus 7 [Mac65]. It has automorphism group $\text{PSL}_6(\mathbb{F}_2)$ (whose order realizes the Hurwitz bound $84(g-1)$ for the maximal number of automorphisms). The $\text{Aut}(C)$ representation $H^0(C, \omega_C)$ must be irreducible (simply because $\text{PSL}(2,8)$ does not have any non-trivial irreducible representation of smaller dimension). By Kempf’s criterion [Kem78 § 5], the syzygy point of $C$ is stable. □

2. Syzygies of tetragonal curves

Let $E$ be the vector bundle $O(3) \oplus O(3) \oplus O(4)$ on $\mathbb{P}^1$. Let $\zeta$ on $\mathbb{P}E$ be the class of $O_{\mathbb{P}E}(1)$. A general tetragonal curve of genus 7 is embedded naturally in $\mathbb{P}E$ via the relative canonical bundle $\omega_C/\mathbb{P}^1$. Its image in $\mathbb{P}E$ is a complete intersection of two surfaces. For a general tetragonal curve, the two surfaces have class $2\zeta - 5f$. There is codimension one sub-locus of the space of tetragonal curves where the two surfaces have classes $2\zeta - 4f$ and $2\zeta - 6f$, respectively. We call the tetragonal curves of the later kind Casnati–Ekedahl special and call their locus the Casnati–Ekedahl locus.

2.1. Generators and relations for the ideal of $\mathbb{P}E$. We first describe the generators and relations for the ideal of $C \subset \mathbb{P}^5$ coming from the scroll $\mathbb{P}E$. Observe that

$$\omega_C = (\zeta - 2f)|_C.$$  

Set $\omega = O(\zeta - 2f)$. We have

$$H^0(\mathbb{P}E, \omega) = H^0(\mathbb{P}^1, O(1) \oplus O(1) \oplus O(2)) = \langle U_1, U_2, V_1, V_2, W_1, W_2, W_3 \rangle.$$

The line bundle $\omega$ embeds $\mathbb{P}E$ as a quartic scroll in $\mathbb{P}^5$. The ideal of $\mathbb{P}E \subset \mathbb{P}^5$ is generated by the six $2 \times 2$ minors of the matrix

$$ \begin{pmatrix} U_1 & V_1 & W_1 & W_2 \\ U_2 & V_2 & W_2 & W_3 \end{pmatrix}. $$

Denote by $Q_{i,j}$ the determinant of the minor formed by columns $i$ and $j$. There are eight syzygies among these quadrics. These are obtained by choosing three of the four columns, one of the two rows,
and using that the $3 \times 3$ matrix formed by the three chosen columns with the chosen row repeated has zero determinant. For example, choosing the first three columns and repeating the first row gives the relation

$$U_1 \cdot Q_{2,3} - V_1 \cdot Q_{1,3} + W_1 \cdot Q_{1,2} = 0.$$ 

There are no further relations among the $Q_{i,j}$. Thus, the scroll $\mathbf{PE}$ accounts for six quadric generators out of ten, and eight syzygies out of sixteen of the canonically embedded curve $C \subset \mathbb{P}^6$.

2.2. Generators and relations for the ideal of $C$. We now describe the remaining quadrics and relations for the ideal of $C$. We must make two cases: one for a general tetragonal curve, and another for a Casnati–Ekedahl special curve. First assume that $C$ is general. Then $I_{C/\mathbf{PE}}$ is generated by two sections, say $\alpha$ and $\beta$, of $O(2\zeta - 5f)$. Observe that

$$O(2\zeta - 5f) = \omega^{\otimes 2} \otimes O(-f).$$

Let $\langle s, t \rangle = H^0(\mathbf{PE}, O(f)) = H^0(\mathbf{P}^1, O(1))$. Then $sa$, $s\beta$, $ta$, and $t\beta$ are the remaining four quadric generators of the canonical ideal of $C$. In other words,

$$I_{C/\mathbf{PE}}(2) = \langle Q_{i,j}, sa, s\beta, ta, t\beta \rangle.$$ 

In addition to the eight syzygies described above, there must be eight additional syzygies among these quadrics. To describe them, let $\xi$ be an element of $H^0(\mathbf{PE}, \omega \otimes O(-f)) = H^0(\mathbf{P}^1, O \oplus O \oplus O(1))$. Each such $\xi$ gives two relations

$$s\xi \otimes t\alpha - t\xi \otimes sa, \quad s\xi \otimes t\beta - t\xi \otimes s\beta.$$ 

Since $\xi$ is chosen from a four dimensional space, we get eight additional syzygies, as required. In summary, the quadrics and syzygies of a general tetragonal curve are given by

\begin{equation}
\begin{aligned}
\text{Quadrics} &= \langle \text{Quadrics } Q_{i,j} \text{ of } \mathbf{PE} \subset \mathbb{P}^6 \rangle \oplus \langle sa, s\beta, ta, t\beta \rangle. \\
\text{Syzygies} &= \langle \text{Syzygies among } Q_{i,j} \rangle \oplus \langle s\xi \otimes t\alpha - t\xi \otimes sa, s\xi \otimes t\beta - t\xi \otimes s\beta \rangle, \\
& \quad \text{where } \xi \in H^0(\mathbf{PE}, O(\zeta - 3f)).
\end{aligned}
\end{equation}

Next, assume that $C$ is Casnati–Ekedahl special. In this case $I_{C/\mathbf{PE}}$ is generated by a section, say $\alpha$, of $O(2\zeta - 4f)$, and a section $\beta$ of $O(2\zeta - 6f)$. Since $O(2\zeta - 4f) = \omega^{\otimes 2}$ and $O(2\zeta - 6f) = \omega^{\otimes 2} \otimes O(-2f)$, the four quadrics in $\mathbf{PE}$ that we get from $\alpha$ and $\beta$ are

$$\alpha, s^2\beta, st\beta, \text{ and } t^2\beta.$$ 

As before, let $\xi$ be an element of $H^0(\mathbf{PE}, \omega(-f))$. Each such $\xi$ gives two relations

$$s\xi \otimes s^2\beta - t\xi \otimes st\beta, \quad s\xi \otimes t^2\beta - t\xi \otimes st\beta.$$ 

Again, with $\xi$ coming from a four dimensional space, we recover the remaining eight syzygies. In summary, the quadrics and syzygies of a Casnati–Ekedahl special tetragonal curve are given by

\begin{equation}
\begin{aligned}
\text{Quadrics} &= \langle \text{Quadrics } Q_{i,j} \text{ of } \mathbf{PE} \subset \mathbb{P}^6 \rangle \oplus \langle \alpha, s^2\beta, st\beta, t^2\beta \rangle. \\
\text{Syzygies} &= \langle \text{Syzygies among } Q_{i,j} \rangle \oplus \langle s\xi \otimes s^2\beta - t\xi \otimes st\beta, s\xi \otimes t^2\beta - t\xi \otimes st\beta \rangle, \\
& \quad \text{where } \xi \in H^0(\mathbf{PE}, O(\zeta - 3f)).
\end{aligned}
\end{equation}

Observe that the syzygies only depend on $\mathbf{PE}$ and $\beta$.

Keep the above setup of a Casnati–Ekedahl special $C$. Let $X \subset \mathbf{PE}$ be the vanishing locus of $\beta$. Recall that $\omega = O_{\mathbf{PE}}(\zeta - 2f)$. 

Proposition 2. The surface $X$ is a del Pezzo surface of degree 6. The line bundle $\omega$ restricts to the anticanonical line bundle of $X$. The ideal of $X \subset \mathbb{P}^6$ is generated by 9 quadrics and the module of syzygies is generated by 16 linear syzygies. In particular, the linear syzygies among the generators of $I_{X/P^6}$ coincides with the linear syzygies among the generators of $I_{C/P^6}$.

Proof. Let $\pi : PE \to \mathbb{P}^1$ be the projection. The relative Euler sequence

$$0 \to \Omega_{PE/P^1} \to \pi^* E \otimes O_{PE}(-1) \to O_{PE} \to 0$$

gives

$$K_{PE} = O_{PE}(-3) \otimes \pi^* \det E \otimes \pi^* K_{P^1} = O_{PE}(-3\zeta + 8f).$$

Since $O_{PE}(X) = O(2\zeta - 6f)$, by adjunction we get

$$K_X = K_{PE} \otimes O_{PE}(X) = O(-\zeta + 2f)|_X = \omega^{-1}|_X.$$

Note that $\omega = \zeta - 2f$ is ample on $PE$. Therefore, $X$ is a surface with ample anti-canonical divisor $\omega|_X$. Using $\zeta^2 = 10\zeta^2$, we compute the degree

$$\omega^2|_X = (\zeta - 2f)^2(2\zeta - 6f) = 6.$$

It is easy to verify that the anti-canonical image of $X \subset P^6$ is cut out by 9 quadrics with 16 linear syzygies. It follows that these must be the 9 quadrics and 16 syzygies in $[2]$ (excluding the quadric $\alpha$). In particular, the linear syzygies among the generators of the ideal of $X$ are the same as those of $C$. □

3. Semistability

Proposition 3. The del Pezzo surface of degree six has a semi-stable first syzygy point in the anti-canonical embedding.

Proof. Let $X$ be the del Pezzo surface. Note that the torus $G_m^2$ acts on $X$. The action makes $H^0(X, -K_X)$ a $G_m^2$ representation. It is easy to check that this representation is multiplicity free. Concretely, we think of $X$ as the blow up of $P^2 = \text{Proj} k[X, Y, Z]$ at the three torus fixed points. Then $H^0(X, -K_X)$ is identified with the cubics passing through these three points, that is, with the vector space

$$\langle X^2Y, XY^2, X^2Z, XZ^2, Y^2Z, YZ^2, X^2Y \rangle.$$

Let the torus act by

$$(s, t) \cdot [X : Y : Z] \to [sX : tY : Z].$$

Then the characters of the sections are given in order by

$$(1, 0), (0, 1), (1, -1), (0, -1), (-1, 1), (-1, 0), (0, 0).$$

By the Kempf–Morisson theorem, the syzygy point of $X$ is $SL_6$ semi-stable if and only if it is $T$ semi-stable, where $T \subset SL_6$ is the maximal torus that acts diagonally on the basis vectors of $H^0(X, -K_X)$ listed above. This torus stability can be checked explicitly by a computer. See delpezzo.m2 and mcsyzygy.m2 for the Macaulay2 code that does this computation. □

Corollary 1. A Casnati–Ekedahl special tetragonal curve of genus 7 has a semi-stable first syzygy point in the canonical embedding.

Proof. The first syzygy point of such curve coincides with the first syzygy point of the del Pezzo surface of degree six. □

Corollary 2. A general tetragonal curve of genus 7 has a semi-stable first syzygy point in the canonical embedding.
Proof. Since semi-stability is an open condition, the assertion follows from the semi-stability of the first syzygy point of a Casnati–Ekedahl special curve.

REFERENCES

